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Quantised fields over de Sitter space

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Abstract. The theory of quantised fields in de Sitter space is investigated. It is demonstrated that de Sitter invariant fields transform like Minkowskian fields under a subgroup of the conformal group, if the de Sitter space is parametrised by horospherical coordinates. Special attention is thus focused on the resolution of the apparent causality conflict. For fields of spin less than or equal to one the field equations are derived; the case of arbitrary spin is dealt with by means of Weinberg-type fields. Furthermore, we give an improved version of the renormalisability proof for the model of a quantised scalar particle in the classical de Sitter background field.

1. Introduction

The de Sitter space is a solution of Einstein's field equation with the cosmological term and vanishing energy–momentum tensor (for a review see, e.g. Weinberg 1972, Hawking and Ellis 1973, Misner *et al* 1973). Its symmetry group contracts into the Poincaré group in the limit of vanishing curvature (Inönü and Wigner 1954). Furthermore, due to the fact that the fixed group of an arbitrary point is isomorphic to the Lorentz group, one can introduce fields over this space in close analogy to Minkowski space.

In view of these attractive properties, the theory of free fields over de Sitter space has already been dealt with repeatedly (Dirac 1935, Gursey 1964, Gutzwiller 1956, Nachtmann 1967, Chernikov and Tagirov 1968, Börner and Dürr 1969, Grensing 1970, Tagirov 1973). Our approach to this subject makes use of results obtained recently in the context of conformal field theory (Rühl 1972, 1973a,b, Grensing 1976). For the parametrisation of de Sitter space by horospherical coordinates (Hannabuss 1971), the transformation law of the fields is shown to be identical with that of Minkowskian fields under a subgroup of the conformal group, which is obtained by setting the time component of translations and special conformal transformations equal to zero and by restricting the Lorentz group to the rotation subgroup. In contrast to conformal invariant fields, the de Sitter fields transform locally under special conformal transformations. However, we do not come into conflict with Einstein causality, which in essence is a consequence of the fact that the de Sitter space can be endowed with an invariant causal structure.

We derive the field equations for particles with spin less than or equal to one by means of a method given in Grensing (1975a); particles of arbitrary spin are discussed for fields of Weinberg type (Weinberg 1964a,b).

It is well known that the de Sitter group admits no positive-definite operator, which could serve as a Hamiltonian (Philips and Wigner 1968). As a substitute, the two independent solutions of the field equations can be chosen such that in the limit of vanishing curvature the positive and negative frequency parts of the corresponding field over Minkowski space are obtained (Nachtmann 1967, Börner and Dürr 1969). An improved version of this limiting process is developed and applied to the case of arbitrary spin.

By means of these results we are able to treat the quantum theory of a scalar field interacting with its classical de Sitter background field. This model has recently been shown to be renormalisable (Candelas and Raine 1975, Dowker and Critchley 1976). We give a rigorous proof of renormalisability, which furthermore makes no use of the Schwinger–De Witt approach.

2. The de Sitter space and its symmetry group

This section presents the necessary preliminaries in a rather condensed form. The reader who is not interested in these technical details can skip to § 3, if he is willing to accept that a field over de Sitter space transforms according to formula (3.2) of the following section.

The de Sitter space can be realised as the sub-manifold†

$$(\xi^0)^2-(\xi^1)^2-(\xi^2)^2-(\xi^3)^2-(\xi^5)^2=-r^2 \tag{2.1}$$

of the pseudo-Euclidean space $\mathbb{R}^{(1,4)}$ with elements $\xi=(\xi^a)_{a=0,1,2,3,5}$ and metric tensor

$$(g_{ab})_{a,b=0,1,2,3,5}=\mathrm{diag}(+1,\,-1,\,-1,\,-1,\,-1). \tag{2.2}$$

Because the hyperboloid of revolution (2.1) is a maximally symmetric space, its symmetry group, the de Sitter group SO(1, 4), has the same dimension as the Poincaré group E(1, 3). The stationary subgroup of the point

$$\mathring{\xi}=(0,0,0,0,-r) \tag{2.3}$$

is the Lorentz group SO(1, 3). Hence the de Sitter space is diffeomorphic to the factor space SO(1, 4)/SO(1, 3). This is to be compared with the well known result that Minkowski space can be identified with the space E(1, 3)/SO(1, 3).

Instead of the restricted de Sitter group, we will later need its spin covering group. The following realisation, denoted by G, is adapted to our purposes (Cartan 1966). An element g of G is supposed to be contained in SL(4, $\mathbb{C}$) and must obey the two conditions

$$g^\dagger E g=E \qquad g^\dagger E' g=E', \tag{2.4}$$

where the matrices E and E' are given by

$$E=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad E'=\begin{pmatrix}\epsilon&0\\0&\epsilon\end{pmatrix}. \tag{2.5}$$

All sub-matrices in (2.5) are two-by-two matrices, and ϵ is anti-symmetric with $\epsilon_{12}=1$. One can show that G is isomorphic to the group Sp(1, 1).

This realisation has the further property that G can be defined as a transformation group on the space $\mathbb{R}^{(1,4)}$. To this end we introduce the matrix

$$\Xi=\xi^a\gamma_a \tag{2.6}$$

† We only discuss the closed de Sitter cosmos. The open model has the unattractive feature that in horospherical coordinates it leads to an uneasy distinction of the third spatial component.

with $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$, where γ^0 is chosen to be

$$\gamma^0=\begin{pmatrix} 0 & -1\\ -1 & 0\end{pmatrix}. \tag{2.7}$$

Note that the γ-matrices $(\gamma^\mu)_{\mu=0,1,2,3}$, enlarged by γ^5, constitute a five-dimensional Clifford algebra on account of

$$\gamma^a\gamma^b+\gamma^b\gamma^a=2g^{ab}. \tag{2.8}$$

Then G acts on $\mathbb{R}^{(1,4)}$ according to

$$\Xi'=g\Xi g^{-1}, \tag{2.9}$$

which gives a homomorphism of G onto the restricted de Sitter group $SO_0(1,4)$ with the explicit form

$$R^a{}_b=\tfrac{1}{4}\mathrm{Tr}(\gamma^a g\gamma_b g^{-1}) \tag{2.10}$$

where $R\in SO_0(1,4)$.

It is a standard task to parametrise G. A convenient choice is

$$\begin{aligned} g&=\begin{pmatrix} 1-\tfrac{1}{2}\hat a & \tfrac{1}{2}\mathrm{i}\hat a\\ \tfrac{1}{2}\mathrm{i}\hat a & 1+\tfrac{1}{2}\hat a\end{pmatrix}\begin{pmatrix}\mathring{A} & 0\\ 0 & \mathring{A}\end{pmatrix}\begin{pmatrix}\cosh\tfrac{1}{2}\lambda & +\mathrm{i}\sinh\tfrac{1}{2}\lambda\\ -\mathrm{i}\sinh\tfrac{1}{2}\lambda & \cosh\tfrac{1}{2}\lambda\end{pmatrix}\begin{pmatrix}1+\tfrac{1}{2}\hat c & \tfrac{1}{2}\mathrm{i}\hat c\\ \tfrac{1}{2}\mathrm{i}\hat c & 1-\tfrac{1}{2}\hat c\end{pmatrix}\\ &=g(\boldsymbol{a})g(\mathring{A})g(d)g(\boldsymbol{c}),\end{aligned} \tag{2.11}$$

where $\mathring{A}\in SU(2)$, $\lambda\in\mathbb{R}$, $\hat a$ is the two-by-two matrix

$$\hat a=\begin{pmatrix} +a^3 & a^1-\mathrm{i}a^2\\ a^1+\mathrm{i}a^2 & -a^3\end{pmatrix}, \tag{2.12}$$

and analogously for $\hat c$. The decomposition (2.11) is not valid globally. However, this can be achieved by making use of the element

$$g_\infty=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \tag{2.13}$$

contained in the Weyl group of G. Furthermore, we remark that the stationary subgroup $\mathring{G}$ of $\mathring{\xi}$ takes the simple form

$$\mathring{g}=\begin{pmatrix}A & \\ & (A^\dagger)^{-1}\end{pmatrix} \tag{2.14}$$

with $A\in SL(2,\mathbb{C})$.

Now we have developed the necessary tools to parametrise the de Sitter hyperboloid. To give a motivation, we note that the product of the first and third factor in (2.11) times a power of the element (2.13) of the Weyl group maps the fixed point into

$$g(\boldsymbol{a})g(d)(g_\infty)^\omega\,.\,\mathring{\xi}=\pm r\begin{pmatrix}+\sinh\lambda-\mathrm{e}^{-\lambda}\tfrac{1}{2}\boldsymbol{a}^2\\ \mathrm{e}^{-\lambda}\boldsymbol{a}\\ -\cosh\lambda+\mathrm{e}^{-\lambda}\tfrac{1}{2}\boldsymbol{a}^2\end{pmatrix} \tag{2.15}$$

with $\omega=0,1$. Thus, we can reach all ξ with $\xi^2=-r^2$ which obey the restriction

$$\xi^0+\xi^5=\mp r\,\mathrm{e}^{-\lambda}\neq 0. \tag{2.16}$$

We introduce the notation

$$z^0 = \pm e^{\lambda}, \qquad z^k = a^k \tag{2.17}$$

and obtain

$$\xi = r\begin{pmatrix} -(1-z^2)/2z_0 \\ z^k/z_0 \\ -(1+z^2)/2z_0 \end{pmatrix}_{k=1,2,3} \tag{2.18}$$

for $z^0 \neq 0$, where $z = (z^{\mu})_{\mu=0,1,2,3}$ is a Minkowskian vector with scalar product $z^2 = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2$. The inversion of (2.18) simply reads

$$z^0 = -r/(\xi^0 + \xi^5), \qquad z^k = -\xi^k/(\xi^0+\xi^5) \tag{2.19}$$

for $\xi^0 + \xi^5 \neq 0$. In these coordinates the fixed point corresponding to $\mathring{\xi}$ is $\mathring{z} = (1, 0, 0, 0)$. If we define

$$g_z = g(\boldsymbol{a})g(d)(g_{\infty})^{\omega} \tag{2.20}$$

with $\boldsymbol{a}$ and λ according to (2.17) and $\omega = (1 - \operatorname{sgn} z^0)/2$ we may write

$$g_z \,.\, \mathring{z} = z. \tag{2.21}$$

This parametrisation of de Sitter space M will be used exclusively in the following. Note that due to $\xi^0 + \xi^5 \neq 0$ there are only two charts which do not overlap and thus constitute no atlas. It is called a horospherical coordinate system because the coordinate hyperplanes $z^0 = \text{constant}$ are imaginary horospheres of the first kind (Gel'fand *et al* 1966). To show this, choose the vector

$$\hat{\xi} = \frac{z^0}{r}(-1, 0, 0, 0, +1)$$

of $\mathbb{R}^{(1,4)}$ with the property $\hat{\xi}^2 = 0$. By virtue of (2.18) ξ then obeys the relation

$$\xi \,.\, \hat{\xi} = 1,$$

which is the defining equation for a horosphere of the first kind. Furthermore, that part of the hyperboloid not covered by the parameters z makes up a horosphere of the second kind. A basic property of these horospherical coordinates is that the intrinsic geometry of the coordinate hyperplanes $z^0 = \text{constant}$ is Euclidean (cf (4.17)). As is concerning the spatial part of z, we thus have a close analogy to Minkowski space.

We conclude this section with a short discussion of the irreducible unitary representations U^{χ} of the quantum mechanical de Sitter group. They are labelled by an index

$$\chi = (l; \Delta), \tag{2.22}$$

where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is an angular momentum variable and Δ is a complex number. For the principal series of representations Δ takes the values

$$\Delta = -\tfrac{3}{2} + \mathrm{i}\rho \tag{2.23}$$

with $\rho \in \mathbb{R}$. It is this series in which we will mainly be interested; the explicit form of these representations is derived in appendix 1. The complete set of inequivalent representations is given in the following list (Hirai 1962, Kuriyan *et al* 1968, Böhm 1973):

Discrete principal series:	$l-2 \geqslant \Delta \geqslant -\frac{1}{2}$	Δ integer or half-integer together with l	
Continuous principal series:	$\Delta = -\frac{3}{2}+i\rho$	$\rho \geqslant 0$	(2.24)
Supplementary series:	$0 > \Delta > -\frac{3}{2}$	$l=0$ and $\Delta \neq -1$	
	$-1 > \Delta > -\frac{3}{2}$	$l \neq 0$ integer	
Exceptional series:	$\Delta = -1$	l integer.	

Actually, the discrete series splits into the analytical and the anti-analytical series. Furthermore, we shall need the eigenvalues of the two Casimir operators of G,

$$C_{\mathrm{II}} = \tfrac{1}{2}M_{ab}M^{ab} \qquad C_{\mathrm{IV}} = W_a W^a \tag{2.25}$$

with

$$W^a = -\tfrac{1}{8}\epsilon^{abcde}M_{bc}M_{de}, \tag{2.26}$$

where by M_{ab} we have denoted the infinitesimal operators of $SO_0(1,4)$ with commutation relations

$$[M_{ab}, M_{cd}] = -i(g_{ac}M_{bd} + g_{bd}M_{ac} - g_{ad}M_{bc} - g_{bc}M_{ad}) \tag{2.27}$$

and ϵ^{abcde} is totally anti-symmetric with $\epsilon^{01235} = 1$. The calculation yields (cf appendix 1):

$$U^{\chi}(C_{\mathrm{II}}) = l(l+1) + \Delta(\Delta+3) \qquad U^{\chi}(C_{\mathrm{IV}}) = l(l+1)(\Delta+2)(\Delta+1). \tag{2.28}$$

Note that the substitution $\Delta \to -3-\Delta$ does not alter the eigenvalues; the representations with label $(l;\Delta)$ and $(l;-3-\Delta)$ can be shown to be equivalent.

3. The transformation law of fields over de Sitter space

We have pointed out that the stationary subgroup $\mathring{G}$ of $\mathring{z}$ is isomorphic to the Lorentz group. Thus we can define fields over de Sitter space and their transformation behaviour in analogy to the flat case by following a well known procedure.

To this end, we choose a finite-dimensional representation D of $SL(2,\mathbb{C})$ over a vector space V and define the representation acting on fields, i.e. differentiable maps $\psi: M \to V$, by

$$T(g)\psi(z) = D(g_z^{-1} g\, g_{g^{-1}.z})\psi(g^{-1}.z). \tag{3.1}$$

There is a profound geometrical concept motivating the introduction of these field representations, which justifies their physical relevance (Varadarajan 1970).

The boost g_z, which enters into (3.1), has been given in (2.20) so that the action of the individual subgroups (2.11) of G may be calculated. Because the factorisation of G is not adapted to the de Sitter space, the little group element $\mathring{g}(z,g) = g_z^{-1} g\, g_{g^{-1}.z}$ must be determined directly; the result is

$$\begin{aligned} &T(\boldsymbol{a})\psi(z) = \psi(z-\boldsymbol{a}) \\ &T(\mathring{A})\psi(z) = D(\mathring{A})\psi(\mathring{\Lambda}^{-1}z) \\ &T(d)\psi(z) = \psi(e^{-\lambda}z) \\ &T(\boldsymbol{c})\psi(z) = D(A(z,\boldsymbol{c}))\psi(z - z^2\boldsymbol{c}/1-2z\,.\,\boldsymbol{c}+z^2\boldsymbol{c}^2) \end{aligned} \tag{3.2}$$

with

$$A(z,\boldsymbol{c}) = 1 + z^0\hat{c} + \hat{z}\hat{c}/(1-2z\,.\,\boldsymbol{c}+z^2\boldsymbol{c}^2)^{1/2}, \tag{3.3}$$

and where we have extended $\boldsymbol{a}$ and $\boldsymbol{c}$ to a four-vector, e.g. $\boldsymbol{c}=(0, c^1, c^2, c^3)$. This transformation law, however, is known from conformal field theory. It coincides precisely with the transformation law of a Minkowskian field with respect to a subgroup of the conformal group (see Grensing 1976, formula (5.1)), which contains pure spatial translations and special conformal transformations, dilatations and rotations. Furthermore, the dimension of the field must be put equal to zero†.

Thus the fields over de Sitter space transform under a subgroup of the conformal group, if the parametrisation by horospherical coordinates is chosen. In particular, a de Sitter field is dilatation invariant and carries dimension zero for arbitrary spin and, as will be shown later, arbitrary mass.

4. The causality problem

Due to the property that the transformation behaviour of a de Sitter invariant field is intimately related to the conformal group, one expects problems with causality.

As to the conformal group, it is known that the causality conflict is avoided for free fields by a non-local transformation law with respect to special conformal transformations (Swieca and Völkel 1973). This has recently been shown to be equally valid for interacting fields (Mack 1975, Grensing 1976).

For the de Sitter group the situation is different, because a de Sitter field transforms locally even under special conformal transformations. We shall prove in the following that a global causal ordering can be defined on de Sitter space, which is invariant under de Sitter transformations. This entails, as will be shown in § 8, that no difficulties occur in connection with Einstein causality.

To begin with, we construct a basis of vector fields on M. At first we choose a canonical basis $\nabla_\mu(\mathring{z})$ for the tangent space over $\mathring{z}$,

$$\nabla_\mu(\mathring{z})=\frac{1}{r}\frac{\partial}{\partial \mathring{z}^\mu}. \tag{4.1}$$

This basis is mapped into a basis of the tangent space over an arbitrary point z with the aid of the boost g_z by defining

$$\nabla_\mu(z)=\lambda_*(g_z)\nabla_\mu(\mathring{z}), \tag{4.2}$$

where $\lambda_*(g)$ is the derived linear function (Brickell and Clark 1970) of the left translation $\lambda(g)z=g\,.\,z$. The explicit form of the vector field (4.2) is found to be

$$\nabla_\mu(z)=\operatorname{sgn}(z^0)\frac{z^0}{r}\frac{\partial}{\partial z^\mu}. \tag{4.3}$$

A simple computation shows that $\mathring{G}$ maps the basis over $\mathring{z}$ into

$$\lambda_*(\mathring{g})\nabla_\mu(\mathring{z})=\Lambda^\nu{}_\mu\nabla_\nu(\mathring{z}), \tag{4.4}$$

where $\Lambda=\tilde{\pi}(\mathring{g})$ with $\tilde{\pi}$ the well known homomorphism of SL(2, C) onto the Lorentz

† On comparing formula (3.2) of this paper with formula (5.1) of Grensing (1976), we omit the ϵ prescription in the latter. We are free to do this, if we do not require the field to be the boundary value of an analytic or anti-analytic function in the forward or backward tube, respectively. Then we obtain a local transformation law even for special conformal transformations.

group. Thus an element of G transforms the vector fields into

$$\lambda_*(g)\nabla_\mu(z) = \Lambda(g\,.\,z, g)^\nu{}_\mu \nabla_\nu(g\,.\,z) \tag{4.5}$$

with $\tilde{\pi}(\mathring{g}(z, g)) = \Lambda(z, g)$ and $\mathring{g}(z, g) = g_z^{-1} g\, g_{g^{-1}.z}$. If the $\nabla_\mu(\mathring{z})$ are normalised according to $(\nabla_\mu(\mathring{z}), \nabla_\nu(\mathring{z})) = g_{\mu\nu}$, then the normalisation of the vector fields is

$$(\nabla_\mu(z), \nabla_\nu(z)) = g_{\mu\nu} \tag{4.6}$$

due to (4.5). This implies that the metric tensor

$$g_{\mu\nu}(z) = \left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right)$$

of de Sitter space reads

$$g_{\mu\nu}(z) = (r/z^0)^2 g_{\mu\nu}, \tag{4.7}$$

which can be shown to be identical with the metric tensor the de Sitter space inherits as a sub-manifold of $\mathbb{R}^{(1,4)}$.

The assertion that the de Sitter space is causally orientable in an invariant manner is an immediate consequence now. We decompose an arbitrary vector field $Z(z)$ over M with respect to the basis,

$$Z(z) = v^\mu(z)\nabla_\mu(z). \tag{4.8}$$

Then the transformed vector field

$$Z'(z') = \lambda_*(g)Z(z) \tag{4.9}$$

with $z' = g\,.\,z$ has components

$$v'^\mu(z') = \Lambda(z', g)^\mu{}_\nu v^\nu(z). \tag{4.10}$$

Hence we can divide the tangent space over an arbitrary point z of M into time-like, light-like, and space-like vectors and define a time orientation in the usual way, which is an invariant characterisation in view of (4.10).

We note that the invariance property of $\operatorname{sgn} v^\mu(z)v_\mu(z)$ and of $\operatorname{sgn} v^0(z)$ for $v^\mu(z)v_\mu(z) \geqslant 0$ can easily be proved directly. The proof is non-trivial only for special conformal transformations, where use has to be made of

$$\frac{\partial z'^\mu}{\partial z^\nu} = \frac{z'^0}{z^0}\bar{g}^\mu{}_\tau(z')\bar{g}^\tau{}_\nu(z) \tag{4.11}$$

with $z' = g^{-1}(\boldsymbol{c})\,.\,z$ and

$$\bar{g}^\mu{}_\nu(z) = g^\mu{}_\nu - 2\frac{z^\mu z_\nu}{z^2}. \tag{4.12}$$

Furthermore, one must take into account that

$$\operatorname{sgn}(1 - 2z\,.\,\boldsymbol{c} + z^2\boldsymbol{c}^2)\bar{g}(z)\bar{g}(z') = \Lambda(z, g(\boldsymbol{c})) \tag{4.13}$$

is an element of the restricted Lorentz group.

For later purposes we determine those points z_2 which are causally related to an arbitrary but fixed point z_1. Thus, given a curve $z(t)$ from $z(0) = z_1$ to $z(1) = z_2$ the vector

$$v^\mu(z(t)) = \operatorname{sgn}(z^0(t))\frac{r}{z^0(t)}\,\dot{z}^\mu(t) \tag{4.14}$$

must obey the conditions $v^\mu(z(t))v_\mu(z(t)) \geq 0$ and $v^0(z(t)) > 0$. Note that the horospherical coordinates are defined for $z^0 \neq 0$ only. Hence we restrict the investigation to $z_1^0, z_2^0 > 0$ at first. If we take as curve $z(t)$ the straight line

$$z(t) = z_1 + t(z_2 - z_1), \tag{4.15}$$

the conditions are met for $(z_2 - z_1)^2 \geq 0$ and $z_2^0 - z_1^0 > 0$. To treat the general case, we compute the distance squared from z_1 to z_2,

$$(s^2)_{12} = \int_0^1 v^\mu(z(t))v_\mu(z(t))\,\mathrm{d}t, \tag{4.16}$$

for the curve (4.15) with the result

$$(s^2)_{12} = \frac{r^2}{z_2^0 z_1^0}(z_2 - z_1)^2. \tag{4.17}$$

Now the condition $(s^2)_{12} \geq 0$ is easily recognised as the generalisation of the condition $(z_2 - z_1)^2 \geq 0$ to arbitrary $z_1^0, z_2^0 \neq 0$ on account of the identity $(s^2)_{12} = (\xi_2 - \xi_1)^2$. Furthermore, from the inequality

$$|\xi_2^0 - \xi_1^0| \geq |\xi_2^5 - \xi_1^5|$$

for $(\xi_2 - \xi_1)^2 \geq 0$ we infer that the sign of $(r/z_1^0) - (r/z_2^0)$ is invariant for $(s^2)_{12} \geq 0$ due to the invariance of $\mathrm{sgn}(\xi_2^0 - \xi_1^0)$ for $(\xi_2 - \xi_1)^2 \geq 0$.

To collect the results, we have shown that z_2 is causally related to z_1 if

$$\frac{r^2}{z_2^0 z_1^0}(z_2 - z_1)^2 \geq 0 \tag{4.18}$$

and

$$\frac{r}{z_2^0 z_1^0}(z_2^0 - z_1^0) > 0, \tag{4.19}$$

and this relation is invariant under de Sitter transformations.

In concluding this section, we give some further geometrical properties of de Sitter space. By means of the metric tensor (4.7) we can calculate the Riemann tensor with the result

$$R_{\mu\nu\rho\tau} = -\frac{1}{r^2}(g_{\mu\rho}(z)g_{\nu\tau}(z) - g_{\mu\tau}(z)g_{\nu\rho}(z)). \tag{4.20}$$

Then the well known definition of the curvature K yields $K = -1/r^2$. Thus, the de Sitter space is a pseudo-Riemannian space of constant negative curvature. Furthermore, the Laplace–Beltrami operator is easily obtained to be

$$\Box(z) = \frac{1}{r^2}\left(z^0 z_0 \frac{\partial}{\partial z^\mu}\frac{\partial}{\partial z_\mu} - 2z^0\frac{\partial}{\partial z^0}\right). \tag{4.21}$$

Finally, we determine the invariant measure over M. To this end, note that the dual basis of differential forms

$$\omega^\mu(z) = \mathrm{sgn}(z^0)(r/z^0)\,\mathrm{d}z^\mu \tag{4.22}$$

transforms according to

$$\lambda^*(g)\omega^\mu(z) = \Lambda(z, g)^\mu{}_\nu\omega^\nu(g^{-1}.z). \tag{4.23}$$

Thus, the measure

$$d\mu(z) = \omega^0(z) \times \omega^1(z) \times \omega^2(z) \times \omega^3(z) = (r/z^0)^4 \, d^4z \tag{4.24}$$

over M is invariant under de Sitter transformations.

5. Contraction

Intuitively it is obvious that the de Sitter hyperboloid goes over into Minkowski space for $r \to \infty$. Furthermore, one then expects the de Sitter group to contract into the Poincaré group in this limit. The contraction process is a delicate limiting procedure and has been dealt with extensively in the literature. In the present case we give a thorough treatment, which heavily relies on the parametrisation by horospherical coordinates.

At first we define the contraction process for de Sitter space. Consider a point ξ with $\xi^2 = -r^2$ in the vicinity of $\mathring{\xi}$. We associate with ξ the point x in the tangent plane at $\mathring{\xi}$ with the coordinates $x^\mu = \xi^\mu$. In the limit $r \to \infty$, the point ξ of de Sitter space coincides with the corresponding point x in the tangent plane,

$$\lim_{r\to\infty} (\xi^0, \boldsymbol{\xi}, \xi^5) = (x^0, \boldsymbol{x}, -\infty). \tag{5.1}$$

Thus, by virtue of (2.19) it is tempting to define variables x by

$$z^0 = -r/(x^0 - r), \qquad z^k = -x^k/(x^0 - r), \tag{5.2}$$

and we expect these coordinates to be adapted to the limiting process $r \to \infty$. Indeed, it is easily shown by means of (5.2) that the basis of vector fields tends to the canonical basis of Minkowski space,

$$\lim_{r\to\infty} \nabla_\mu(z) = \frac{\partial}{\partial x^\mu}, \tag{5.3}$$

and the line element into the Minkowskian value

$$\lim_{r\to\infty} g_{\mu\nu}(z) \, \mathrm{d}z^\mu \, \mathrm{d}z^\nu = g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu. \tag{5.4}$$

Furthermore, the limit of the Laplace–Beltrami operator is the Klein–Gordon operator,

$$\lim_{r\to\infty} \Box(z) = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu}. \tag{5.5}$$

We now turn to the contraction process for the de Sitter group. It is evident that rotations with parameters $\alpha^{\mu\nu}$ will correspond to Lorentz transformations Λ in the tangent plane, and that rotations with parameters $\alpha^{\mu 5}$ will go over into translations a^μ, if $\alpha^{\mu 5} \to 0$ for $r \to \infty$ such that the limit

$$\lim_{r\to\infty} r\alpha^{\mu 5} = a^\mu \tag{5.6}$$

is finite. To investigate this in detail, we need the parametrisation of G in terms of the parameters $\alpha^{ab} = -\alpha^{ba}$, which may be shown to read $g = g'\mathring{g}$ with $\mathring{g}$ according to (2.14) and

$$g' = \begin{pmatrix} \cosh\frac{1}{2}\lambda & +\mathrm{i}\sinh\frac{1}{2}\lambda \\ -\mathrm{i}\sinh\frac{1}{2}\lambda & \cosh\frac{1}{2}\lambda \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} \mathring{A} + \mathring{A}^\dagger & \mathring{A} - \mathring{A}^\dagger \\ \mathring{A} - \mathring{A}^\dagger & \mathring{A} + \mathring{A}^\dagger \end{pmatrix} \tag{5.7}$$

where $\lambda \in \mathbb{R}$ and $\mathring{A} \in SU(2)$. We define $\lambda = \alpha^{05}$ and $\alpha^k = \alpha^{k5}$ with α^k the parameters of $\mathring{A}$, so that g' acts on ξ as usual. Then, by taking into account that

$$x^0 = \lim_{r\to\infty} r(z^0 - 1), \qquad x^k = \lim_{r\to\infty} \frac{r}{z^0} z^k, \tag{5.8}$$

a simple calculation yields

$$\lim_{r\to\infty} r(z'^0 - 1) = \Lambda^0{}_\nu x^\nu + a^0, \qquad \lim_{r\to\infty} \frac{r}{z'^0} z'^k = \Lambda^k{}_\nu x^\nu + a^k \tag{5.9}$$

for $z' = g \,.\, z$, which is the well known action of the Poincaré group. It is now obvious from (5.6) and (5.9), and will be shown explicitly in the next section, that the limit of $M_{\mu\nu}$ and $-M_{\mu 5}/r$ for $r \to \infty$ is finite and yields the infinitesimal operators of the Poincaré group. We thus define

$$P_\mu = -\frac{1}{r} M_{\mu 5} \tag{5.10}$$

and obtain the commutation relations

$$\begin{aligned} &[M_{\mu\nu}, P_\tau] = -\mathrm{i}(g_{\mu\tau} P_\nu - g_{\nu\tau} P_\mu) \\ &[P_\mu, P_\nu] = \frac{\mathrm{i}}{r^2} M_{\mu\nu} \to 0. \end{aligned} \tag{5.11}$$

Furthermore, we discuss the contraction process for the field representations. As we shall prove in § 8 the limit

$$\lim_{r\to\infty} r^{-1} \psi(z) = \phi(x)$$

for the de Sitter field exists at least on a certain sub-space of functions. If we assume this to be true in general, then it easily follows by means of (5.9) and $\lim_{r\to\infty} g_z = e$ that

$$\lim_{r\to\infty} r^{-1} T(g) \psi(z) = D(A) \phi(\Lambda^{-1}(x - a)), \tag{5.12}$$

which is the usual transformation law of a relativistic field in flat space.

The contraction process for the representations of the continuous principal series has been investigated in the literature (Mickelsson and Niederle 1972, Böhm 1973). We only make some qualitative remarks on this point. As is evident from the foregoing discussion, the Casimir operators multiplied by $1/r^2$ have a finite limit for $r \to \infty$. They contract into

$$\lim_{r\to\infty} \frac{1}{r^2} C_{\mathrm{II}} = -P_\mu P^\mu \qquad \lim_{r\to\infty} \frac{1}{r^2} C_{\mathrm{IV}} = W_\mu W^\mu \tag{5.13}$$

with $W^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\tau} M_{\nu\rho} P_\tau$, which are the Casimir operators of the Poincaré group. Correspondingly, the eigenvalues (2.28) of the Casimir operators have a finite limit, if $\rho \to \infty$ for $r \to \infty$ such that

$$\lim_{r\to\infty} \rho/r = m \tag{5.14}$$

is finite. Then we obtain

$$\lim_{r\to\infty} \frac{1}{r^2} U^\chi(C_{\mathrm{II}}) = -m^2 \qquad \lim_{r\to\infty} \frac{1}{r^2} U^\chi(C_{\mathrm{IV}}) = -m^2 l(l+1), \tag{5.15}$$

which are the eigenvalues of the Casimir operators of the Poincaré group for real mass representations. Note, however, that the contracted representation $\chi=(l;\ -\frac{3}{2}+\mathrm{i}\rho)$ is no longer irreducible. It can be shown to be the direct sum of the representations with positive and negative energy.

6. Field equations

The field representations over de Sitter space are neither irreducible nor unitary; they are reducible even if the inducing representation is irreducible. In the Minkowskian case the field equations are known to enforce irreducibility and unitarity (Niederer and O'Raifertaigh 1974, Grensing 1975a). A genuine and constructive approach to derive the field equations over flat space has been given in Grensing (1975a), which can be applied analogously to de Sitter space (Grensing 1970). In part this method has also been used in Börner and Dürr (1969).

The infinitesimal operators of the field representation can be computed from (3.2) to yield

$$\begin{aligned}
T(P_0)&=\mathrm{i}z^{\mu}\partial_{\mu}/r\\
T(P_k)&=-(\Sigma_{k\mu}z^{\mu}+\mathrm{i}z_kz_{\mu}\partial^{\mu}-\tfrac{1}{2}\mathrm{i}(1+z^2)\,\partial_k)/r\\
T(M_{k0})&=\Sigma_{k\mu}z^{\mu}+\mathrm{i}z_kz_{\mu}\partial^{\mu}+\tfrac{1}{2}\mathrm{i}(1-z^2)\partial_k\\
T(M_{ki})&=\Sigma_{ki}+\mathrm{i}(z_k\partial_i-z_i\partial_k),
\end{aligned}\tag{6.1}$$

where the $\Sigma_{\mu\nu}$ denote the infinitesimal operators of D. A simple calculation shows that they indeed contract into the Poincaré operators

$$\lim_{r\to\infty}T(P_{\mu})=\mathrm{i}\frac{\partial}{\partial x^{\mu}}\qquad \lim_{r\to\infty}T(M_{\mu\nu})=\Sigma_{\mu\nu}+\mathrm{i}\left(x_{\mu}\frac{\partial}{\partial x^{\nu}}-x_{\nu}\frac{\partial}{\partial x^{\mu}}\right),\tag{6.2}$$

a result which has been anticipated in § 5 for the scalar case.

The Lie algebra operators (6.1) act as differential operators. Hence the Casimir operators supply us with differential equations, if we require $T(C_{\mathrm{II}})$ and $T(C_{\mathrm{IV}})$ to be a multiple of the unit operator with eigenvalues (2.28). These conditions restrict the function space to an invariant sub-space.

Thus, in the case of the Casimir operator of second order, the field is subjected to the condition

$$(r^2\Box(z)+\tfrac{1}{2}\Sigma_{ik}\Sigma^{ik}-2\mathrm{i}\Sigma_{0k}z^0\partial^k)\psi(z)=[l(l+1)+\Delta(\Delta+3)]\psi(z),\tag{6.3}$$

which replaces the usual Klein–Gordon equation. Note that (6.3) depends on the inducing representation in contrast to the flat case.

For the Casimir operator of fourth order we must specify the representation D of the Lorentz group. Furthermore, the discrete transformations must be taken into account. They have been dealt with in detail in appendix 2. We will investigate only three cases, for which the inducing representation D and the representation of the discrete transformations have the following form:

$$s=0:\qquad D(\mathring{g})=1$$

$$T(g_{\mathrm{P}})\psi(z)=\hat{\epsilon}_{\mathrm{P}}\psi(\Lambda_{\mathrm{P}}^{-1}z),\qquad T(g_{\mathrm{T}})\psi(z)=\hat{\epsilon}_{\mathrm{T}}\psi(\Lambda_{\mathrm{T}}^{-1}z)\tag{6.4}$$

$s=\frac{1}{2}$: $\quad D(\mathring{g})=\mathring{g}$

$$T(g_P)\psi(z)=\mathrm{sgn}(z^0)\gamma^0\psi(\Lambda_P^{-1}z), \qquad T(g_T)\psi(z)=\mathrm{i}\hat{\epsilon}_T\gamma^0\gamma^5\psi(\Lambda_T^{-1}z) \tag{6.5}$$

$s=1$: $\quad D(\mathring{g})=\Lambda$

$$T(g_P)\psi(z)=\hat{\epsilon}_P\Lambda_P\psi(\Lambda_P^{-1}z), \qquad T(g_T)\psi(z)=\hat{\epsilon}_T\Lambda_T\psi(\Lambda_T^{-1}z). \tag{6.6}$$

In (6.4)–(6.6) we have denoted by P and T the space and time inversion. Their representations contract into the most frequently used in flat space. Furthermore, the total inversion is given as usual by $g_{PT}=g_P g_T$ and $\hat{\epsilon}_P$, $\hat{\epsilon}_T$ take values ± 1. Note that for $s=\frac{1}{2}$ the discrete transformations commute, though the contracted transformations anti-commute.

Now we are ready to determine the conditions imposed by the invariant operator of fourth order (see equation (A.10)).

$s=0$: $\quad T(C_{IV})=0.$

On comparing with (2.28) we obtain $l=0$.

$s=\frac{1}{2}$: $\quad T(C_{IV})=\frac{3}{4}T(C_{II})+\frac{15}{16}.$

Analogously, this condition requires $l=\frac{1}{2}$.

$s=1$: $\quad (T(C_{IV})\psi)^\nu(z)=2(T(C_{II})\psi)^\nu(z)+2z_0\partial^\nu[3\psi^0(z)-z^0\partial_\mu\psi^\mu(z)].$

In order to make $T(C_{IV})$ a multiple of the identity operator, we set the second term on the right-hand side equal to zero. This is achieved by (Grensing 1970, Börner 1971)

$$3\psi^0(z)-z^0\partial_\mu\psi^\mu(z)=0, \tag{6.7}$$

which is the Lorentz condition in de Sitter space. Furthermore we obtain $l=1$.

With that we have concluded the investigation of the restrictions implied by the Casimir operators. However, for $s=\frac{1}{2}$ the field representation is still reducible. Consequently we try to find a further invariant operator, which for reasons of simplicity we take to be a differential operator of first order,

$$D(z)=C^\mu(z)\partial_\mu+C(z). \tag{6.8}$$

Then the invariance conditions

$$[T(P_\mu),D(z)]=0=[T(M_{\mu\nu}),D(z)] \tag{6.9}$$

determine the functional dependence on z,

$$C^\mu(z)=z^0C^\mu, \qquad C(z)=C, \tag{6.10}$$

and the constant four-by-four matrices C^μ and C must obey

$$[\Sigma_{\mu\nu},C_\tau]=-\mathrm{i}(g_{\mu\tau}C_\nu-g_{\nu\tau}C_\mu) \tag{6.11}$$

$$[\Sigma_{ik},C]=0, \qquad [\Sigma_{i0},C]=C^\mu\Sigma_{i\mu}. \tag{6.12}$$

Equation (6.11) can obviously be satisfied with $C^\mu=\gamma^\mu$, and from (6.12) we obtain $C=-\frac{3}{2}\gamma^0$. Thus the invariant operator is determined to be

$$D(z)=z^0\gamma^\mu\partial_\mu-\tfrac{3}{2}\gamma^0. \tag{6.13}$$

The eigenvalue of $D(z)$ is found by observing that

$$D^2(z)=T(C_{II})+\tfrac{3}{2}=-\rho^2,$$

hence we require $D(z) = \pm i\rho$. We take the plus sign due to our unconventional choice of γ^0 and obtain

$$(z^0\gamma^\mu\partial_\mu - \tfrac{3}{2}\gamma^0)\psi(z) = i\rho\psi(z), \tag{6.14}$$

which is the Dirac equation in de Sitter space (Nachtmann 1967).

It now remains to examine the compatability with the discrete transformations. As is easily checked, the parity operator commutes with $D(z)$. However, this does not hold for the time inversion. Hence the time inversion is no symmetry of the theory—at least for our choice (6.5) of discrete transformations. Therefore we omit the time inversion in the following investigations, though the field equations for $s = 0$ and $s = 1$ are invariant under all discrete transformations.

We finally remark that the field equations contract into the corresponding field equations of Minkowski space. For the Klein–Gordon equation (6.3) this is obvious from the results of § 5, and for the Lorentz condition (6.7) and the Dirac equation (6.14) the assertion follows upon multiplication with $1/r$. It is thus tempting to interpret them as field equations for particles with spin $s = 0, \frac{1}{2}, 1$ and mass $m = \rho/r$. Further support for this interpretation will be given in the following.

7. Generalised covariant derivative

In § 4 we have introduced the basis (4.3) of vector fields over de Sitter space. By definition, they act on scalar functions ψ over M. These vector fields have the basic transformation property (4.5), which we rewrite by using the notation $\lambda_*^{-1}(g) = T_*(g)$ as follows,

$$T_*(g)\nabla^\mu(z)\psi(z) = \nabla^\mu(z)T(g)\psi(z) = \Lambda(z, g)^\mu{}_\nu\nabla^\nu(g^{-1}.z)\psi(g^{-1}.z) \tag{7.1}$$

with $\nabla^\mu(z) = g^{\mu\nu}\nabla_\nu(z)$ and $T(g)$ the scalar field representation. Thus, $T_*(g)$ is a field representation that is obtained by commuting $\nabla(z) = (\nabla^\mu(z))_{\mu=0,1,2,3}$ with $T(g)$,

$$T_*(g)\nabla(z) = \nabla(z)T(g), \tag{7.2}$$

and is induced by the self-representation of the Lorentz group.

We want to generalise $\nabla(z)$ such that (7.2) remains valid for arbitrary field representations $T(g)$. The explicit form of (7.2) will then read

$$\nabla^\mu(z)D^A{}_B(\mathring{g}(z, g))\psi^B(g^{-1}.z) = \Lambda^\mu{}_\nu(z, g)D^A{}_B(\mathring{g}(z, g))\nabla^\nu(g^{-1}.z)\psi^B(g^{-1}.z) \tag{7.3}$$

with $A, B = 1, \ldots, N$ and N the dimension of D. Hence the 'derived induced representation' $T_*(g)$ is induced by the representation $\Lambda\otimes D(\mathring{g})$ and acts on $\nabla(z)\psi(z)$ according to (3.1).

The generalisation of the vector fields $\nabla_\mu(z)$ is obtained by adding to (4.3) a z-dependent $N\times N$ matrix,

$$\nabla_\mu(z) = e^\nu{}_\mu(z)\left(\frac{\partial}{\partial z^\nu} + \Gamma_\nu(z)\right), \tag{7.4}$$

where we have introduced the inverse *vierbein*

$$e^\nu{}_\mu(z) = \frac{z^0}{r}g^\nu{}_\mu. \tag{7.5}$$

Due to (7.3), the $\Gamma_\nu(z)$ must obey

$$D(\mathring{g}(z,g))^{-1}\Big(\frac{\partial}{\partial z^\nu}+\Gamma_\nu(z)\Big)D(\mathring{g}(z,g))=\frac{\partial z'^\mu}{\partial z^\nu}\Gamma_\mu(z') \tag{7.6}$$

with $z'=g^{-1}\,.\,z$. For the individual sub-groups of the factorisation (2.11) the condition (7.6) splits into

$$\Gamma_\mu(z)=\Gamma_\mu(z-\boldsymbol{a}) \tag{7.7}$$

$$D^{-1}(\mathring{A})\Gamma_\mu(z)D(\mathring{A})=\mathring{\Lambda}_\mu{}^\nu\Gamma_\nu(\mathring{\Lambda}^{-1}z) \tag{7.8}$$

$$\mathrm{e}^\lambda\Gamma_\mu(z)=\Gamma_\mu(\mathrm{e}^{-\lambda}z) \tag{7.9}$$

$$D^{-1}(A(z,\boldsymbol{c}))(\partial_\mu+\Gamma_\mu(z))D(A(z,\boldsymbol{c}))=\frac{z'^0}{z^0}\bar{g}(z')^\nu{}_\tau\bar{g}(z)^\tau{}_\mu\Gamma_\nu(z') \tag{7.10}$$

$$z'=z-z^2\boldsymbol{c}/1-2z\,.\,\boldsymbol{c}+z^2\boldsymbol{c}^2,$$

where we have used (4.11). Equation (7.7) requires $\Gamma_\mu(z)$ to be a function of z^0 only, and (7.9) shows that

$$\Gamma_\mu(z)=\Gamma_\mu/z^0 \tag{7.11}$$

with the Γ_μ being constant $N\times N$ matrices. To exploit (7.10), we restrict to $|\boldsymbol{c}|\ll 1$ and get

$$\boldsymbol{c}^\nu\Sigma_{\nu\mu}+\frac{\mathrm{i}}{z^0}(\boldsymbol{c}^\nu z_\mu-z^\nu\boldsymbol{c}_\mu)\Gamma_\nu=\frac{1}{z^0}\boldsymbol{c}^\nu z^\tau[\Sigma_{\nu\tau},\Gamma_\mu]$$

by means of (4.13). Combined with (7.8), this condition yields

$$\Sigma_{i0}+\mathrm{i}\Gamma_i=[\Sigma_{i0},\Gamma_0]$$

$$\Sigma_{ik}=[\Sigma_{i0},\Gamma_k]$$

so that

$$\Gamma_\mu=\mathrm{i}\Sigma_{\mu 0}. \tag{7.12}$$

Hence we obtain†

$$\nabla_\mu(z)=\frac{1}{r}(z_0\partial_\mu-\mathrm{i}\Sigma_{0\mu}) \tag{7.13}$$

as the final result.

By means of (7.13) we can now define the tensor–spinor

$$\psi^{\mu_1\ldots\mu_r A}(z)=\nabla^{\mu_1}(z)\ldots\nabla^{\mu_r}(z)\psi^A(z), \tag{7.14}$$

which thus transforms under the field representation induced by the representation $\overset{r}{\otimes}\Lambda\otimes D(A)$. This is the basic property of the generalised covariant derivative.

† This form of a generalised covariant derivative has been given already by Nachtmann (1967). Note, however, that his method is not capable of yielding the second term on the right-hand side of (7.13). This is a consequence of the fact that Nachtmann's definition (2.18) coincides with our definition (4.2), as follows from

$$\nabla_\mu(z)\psi(z)=\frac{\partial}{\partial z'^\mu}T(g_z)^{-1}\psi(z')|_{z'=\mathring{z}}=\frac{\partial}{\partial z'^\mu}D(\mathring{g}(z',g_z))\psi(g_z\,.\,z')|_{z'=\mathring{z}}$$

and $\mathring{g}(z',g_z)=e$.

In connection with these results we make some further comments on the field equations. The operator of second order $g^{\mu\nu}\nabla_\mu(z)\nabla_\nu(z)$ is calculated to be

$$r^2\nabla^\mu(z)\nabla_\mu(z) = T(C_{\mathrm{II}}) - \tfrac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}, \tag{7.15}$$

so that for the field representations in question the operator (7.15) is a multiple of the identity simultaneously with $T(C_{\mathrm{II}})$. The Dirac equation (6.14) takes the suggestive form

$$(\mathrm{i}\gamma^\mu\nabla_\mu(z) + m)\psi(z) = 0 \tag{7.16}$$

with $m = \rho/r$, which may be used to yield an independent proof of its invariance. Furthermore, the Lorentz condition (6.7) now reads

$$(\nabla_\mu(z)\psi)^\mu(z) = 0. \tag{7.17}$$

Its role is further clarified by noting that Gauss' theorem with respect to the differential forms

$$\Sigma^\mu(z) = -\tfrac{1}{6}\epsilon^{\mu\nu\rho\tau}\omega_\nu(z)\times\omega_\rho(z)\times\omega_\tau(z) \tag{7.18}$$

and for an arbitrary vector field ψ^μ is

$$\int_{\partial\Sigma} \Sigma_\mu(z)\psi^\mu(z) = \int_\Sigma \mathrm{d}\mu(z)\nabla_\mu(z)\psi^\mu(z). \tag{7.19}$$

The usual form of this integral theorem is obtained, if we pass from ψ^μ to

$$A^\mu(z) = e^\mu{}_\nu(z)\psi^\nu(z) \tag{7.20}$$

and take into account that

$$\frac{1}{r}\nabla_\nu(z)\psi^\mu(z) = \partial_\nu A^\mu(z) + \Gamma^\mu{}_{\nu\tau}(z)A^\tau(z), \tag{7.21}$$

where $\Gamma^\mu{}_{\nu\tau}(z)$ is the Christoffel symbol. It is readily shown that A^μ is a vector in the sense of Riemannian geometry. This property together with (7.21) makes explicit the relation between the generalised and the ordinary covariant derivative.

8. Solutions of the field equations and Einstein causality

Having derived the field equations in the preceding sections, we now give a systematic treatment of their solutions.

At first we study the case of arbitrary spin s† for fields of type $(s, 0)\oplus(0, s)$ *à la* Weinberg. The only field equation is

$$(r^2\Box(z) - 2\mathrm{i}\Sigma_{0k}z^0\partial^k)\psi(z) = \Delta(\Delta+3)\psi(z), \tag{8.1}$$

where the $\Sigma_{\mu\nu}$ are the infinitesimal operators of the representation $D(A) = D^{(s)}(A)$ $\oplus D^{(s)}(A)^{\dagger-1}$ acting on $\psi = \chi\oplus\dot{\chi}$. Furthermore, $D^{(s)}(A)$ and $D^{(s)}(A)^{\dagger-1}$ are the well known representations $(s, 0)$ and $(0, s)$ of the Lorentz group (for the explicit definition,

† In Mielke (1977), the generalisation to arbitrary spin has been tried by means of Bargmann–Wigner equations. However, the equations given there are not even invariant. Furthermore, it is erroneously believed that the operator $\frac{1}{2}\Sigma_{ik}\Sigma^{ik}$ acts on the Bargmann–Wigner field as a multiple of the identity with eigenvalue $s(s+1)$, which is known to be true for $s \leq \frac{1}{2}$ only.

see, e.g. Grensing 1976, formula (B.22)). The differential equation (8.1) decouples into separate equations for the $(2s+1)$-component fields χ and $\dot{\chi}$, which have two independent solutions denoted by $\chi^{\pm}$ and $\dot{\chi}^{\pm}$. We call them positive and negative frequency solutions for reasons which will become clear shortly. On performing a Fourier transformation with respect to the spatial part of z, e.g.

$$\chi^{\pm}(z)=\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}\,\mathrm{e}^{\pm \mathrm{i}\boldsymbol{p}\boldsymbol{z}}\chi^{\pm}(z^0,\boldsymbol{p}), \tag{8.2}$$

we obtain

$$\left(z_0^2\frac{\mathrm{d}^2}{\mathrm{d}z_0^2}-2z_0\frac{\mathrm{d}}{\mathrm{d}z_0}+(\boldsymbol{p}z_0)^2\mp 2D^{(s)}(M_{0k})z^0p^k-\Delta(\Delta+3)\right)\chi^{\pm}(z^0,\boldsymbol{p})=0.$$

This set of coupled differential equations can be reduced by choosing a helicity basis,

$$\chi^{+}(z^0,\boldsymbol{p})=\sum_{s_3=-s}^{s} a^{+}(z^0,\boldsymbol{p};s_3)\chi(\boldsymbol{p},s_3),\qquad \chi^{-}(z^0,\boldsymbol{p})=\sum_{s_3=-s}^{s} c^{-}(z^0,\boldsymbol{p};s_3)\chi(\boldsymbol{p},s_3), \tag{8.3}$$

where a^+ and c^- are scalar functions and $\chi(\boldsymbol{p},s_3)$ is defined by

$$\chi(\boldsymbol{p},s_3)=(D^{(s)}(\mathring{A}_{\boldsymbol{p}})_{s_3's_3})_{s_3'=-s,\ldots,+s} \tag{8.4}$$

with

$$\mathring{A}_{\boldsymbol{p}}=\frac{1}{\sqrt{[2p(p+p^3)]}}\begin{pmatrix} p+p^3 & -p^1+\mathrm{i}p^2 \\ +p^1+\mathrm{i}p^2 & p+p^3\end{pmatrix} \tag{8.5}$$

and $p=|\boldsymbol{p}|$. The rotation (8.5) maps $(0,0,p)$ into $\boldsymbol{p}=(p^1,p^2,p^3)$, which is used to show that the basis functions $\chi(\boldsymbol{p},s_3)$ have the property

$$\frac{1}{p}\boldsymbol{p}\cdot D^{(s)}(\boldsymbol{M})\chi(\boldsymbol{p},s_3)=s_3\chi(\boldsymbol{p},s_3), \tag{8.6}$$

where $M^{12}=M^3$ etc. Then, by taking into account that

$$+\mathrm{i}D^{(s,0)}(M^{k0})=D^{(s)}(M^k)=-\mathrm{i}D^{(0,s)}(M^{ko}), \tag{8.7}$$

we end up with a differential equation for a^+ and c^-, e.g.

$$\left(w^2\frac{\mathrm{d}^2}{\mathrm{d}w^2}-2w\frac{\mathrm{d}}{\mathrm{d}w}+w^2+2\mathrm{i}s_3w-\Delta(\Delta+3)\right)a^+(z^0,\boldsymbol{p};s_3)=0$$

with $w=pz^0$. The solutions are

$$\begin{aligned} a^+(z^0,\boldsymbol{p};s_3)&=a^+(\boldsymbol{p},s_3)N^*(p,\rho)wM^*_{s_3,\mathrm{i}\rho}(-2\mathrm{i}w)\\ c^-(z^0,\boldsymbol{p};s_3)&=c^-(\boldsymbol{p},s_3)N(p,\rho)wM_{s_3,\mathrm{i}\rho}(-2\mathrm{i}w),\end{aligned} \tag{8.8}$$

where $M_{s_3,\mathrm{i}\rho}(-2\mathrm{i}\omega)$ is a Whittaker function (Magnus *et al* 1966), $a^+(p,s_3)$ and $c^-(p,s_3)$ are arbitrary Fourier amplitudes, and $N(p,\rho)$ is a normalisation factor. The corresponding functions $\dot{\chi}^{\pm}$ are obtained by simply replacing the index s_3 of the Whittaker functions by its negative due to (8.7). The solutions of (8.1) are built from the solutions χ and $\dot{\chi}$ and read

$$\psi(z)=\sum_{s_3}\int\frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}(\mathrm{e}^{+\mathrm{i}\boldsymbol{p}\boldsymbol{z}}a^+(\boldsymbol{p},s_3)\psi^+(z^0,\boldsymbol{p};s_3)+\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\boldsymbol{z}}c^-(\boldsymbol{p},s_3)\psi^-(z^0,\boldsymbol{p};s_3)), \tag{8.9}$$

with the fundamental solutions being given by

$$\psi^+(z^0, \boldsymbol{p}; s_3) = N^*(p,\rho)\frac{1}{\sqrt{2}}\begin{pmatrix} wM^*_{+s_3,\mathrm{i}\rho}(-2\mathrm{i}w)\chi(\boldsymbol{p}, s_3) \\ wM^*_{-s_3,\mathrm{i}\rho}(-2\mathrm{i}w)\chi(\boldsymbol{p}, s_3)\end{pmatrix}$$
$$\psi^-(z^0, \boldsymbol{p}; s_3) = N(p,\rho)\frac{1}{\sqrt{2}}\begin{pmatrix} wM_{+s_3,\mathrm{i}\rho}(-2\mathrm{i}w)\chi(\boldsymbol{p}, s_3) \\ (-1)^{2s}wM_{-s_3,\mathrm{i}\rho}(-2\mathrm{i}w)\chi(\boldsymbol{p}, s_3)\end{pmatrix}. \tag{8.10}$$

The factor $(-1)^{2s}$ in front of the lower components of $\psi^-(z^0, \boldsymbol{p}; s_3)$ has been chosen in analogy to the flat case (Weinberg 1964a,b).

Now we are left with the determination of the normalisation factor $N(p,\rho)$, which is done as follows. The Hermitian form

$$(\psi_1, \psi_2) = \mathrm{i}r^{-2}\int_{z^0=\mathrm{constant}} \Sigma^\mu(z)[\bar\psi_1^+(z)\overleftrightarrow{\nabla}_\mu(z)\psi_2^+(z) - (-1)^{2s}\bar\psi_1^-(z)\overleftrightarrow{\nabla}_\mu(z)\psi_2^-(z)], \tag{8.11}$$

where by definition

$$\bar\psi_1(z)\overleftrightarrow{\nabla}_\mu(z)\psi_2(z) = \bar\psi_1(z)(\nabla_\mu(z)\psi_2(z)) - (\overline{\nabla_\mu(z)\psi_1(z)})\psi_2(z)$$

and $\bar\psi = (\dot\chi^+, \chi^+)$, is invariant under de Sitter transformations (3.1) due to (4.23) and (7.3). Furthermore, it is independent of z^0 on account of the identity

$$\nabla^\mu(z)(\bar\psi_1^\pm(z)\overleftrightarrow{\nabla}_\mu(z)\psi_2^\pm(z)) = 0$$

and Gauss' theorem (7.19). Note that a factor r^{-2} has been inserted in (8.11) to give the right-hand side the dimension zero. We fix $N(p,\rho)$ by requiring that (8.11) takes the form

$$(\psi_1, \psi_2) = \sum_{s_3}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^3}(a_1^+(\boldsymbol{p}, s_3)^* a_2^+(\boldsymbol{p}, s_3) + c_1^-(\boldsymbol{p}, s_3)^* c_2^-(\boldsymbol{p}, s_3)) \tag{8.12}$$

in Fourier space, which yields

$$N(p,\rho) = \mathrm{e}^{\mathrm{i}\theta(p,\rho)}\,\mathrm{e}^{-\mathrm{i}\pi\rho/2}/\sqrt{(4\rho p^3)} \tag{8.13}$$

with $\mathrm{e}^{\mathrm{i}\theta(p,\rho)}$ an arbitrary phase factor†. In addition we have proved that (8.11) is an inner product, with respect to which the field representation (3.1) is unitary.

As is obvious from (8.13), these investigations are valid for $\rho > 0$ only. We will make some remarks on the existence of mass zero particles in de Sitter space in § 10.

We next turn to the investigation of the contraction process. We expect the limit

$$\phi(x) = \lim_{r\to\infty}\sqrt{(2)}m^s\frac{1}{r}\psi(z), \tag{8.14}$$

with x and z being related by (5.2), to be finite because the scalar product (8.11) then contracts into the well known scalar product

$$(\phi_1, \phi_2) = \frac{1}{2m^{2s}}\mathrm{i}\int \mathrm{d}^3x\left(\bar\phi_1^+(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}\phi_2^+(x) - (-1)^{2s}\bar\phi_1^-(x)\frac{\overleftrightarrow{\partial}}{\partial x^0}\phi_2^-(x)\right)$$

† Börner and Dürr (1969) have normalised the solutions with respect to the invariant measure (4.24). However, then the scalar product does not contract into the corresponding scalar product of Minkowski space.

in Minkowski space. Furthermore, we define, e.g.

$$a^+(k, s_3) = \frac{1}{\sqrt{(2)}m^s} r\sqrt{(2\omega(p))}a^+(\boldsymbol{p}, s_3) \tag{8.15}$$

with $\boldsymbol{p} = r\boldsymbol{k}$ and $\omega(p) = \sqrt{(\rho^2 + \boldsymbol{p}^2)}$ so that (8.12) goes over into the usual Minkowskian expression

$$(\phi_1, \phi_2) = 2m^{2s} \sum_{s_3} \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(k)} (a_1^+(k, s_3)^* a_2^+(k, s_3) + c_1^-(k, s_3)^* c_2^-(k, s_3))$$
$$\omega(k) = \sqrt{(m^2 + \boldsymbol{k}^2)}.$$

Note that the field ϕ carries the correct canonical dimension $\delta = s + 1$ due to (8.14) and (8.15). In order to compute the limit (8.14), we use the following asymptotic expansion of the Whittaker functions $M_{k,m}(z)$ for large $|m|$ and $|z|$ (see Kazarinoff 1955, as cited in Magnus *et al* 1966, p 319),

$$M_{k,m}(2mz) \sim 2^{2m+1/2} m^{m+1/2} z^{-m+1/2} [(z^2+1)^{1/2} - 1]^m$$
$$\times [(z^2+1)^{1/2} - z]^k (z^2+1)^{-1/4} \exp\{m[(z^2+1)^{1/2} - 1]\} \tag{8.16}$$
$$|z/2m| < 1. \tag{8.17}$$

The inequality (8.17) applies to our case since we have to investigate $M_{s_3,i\rho}(-2ir|\boldsymbol{k}|z^0)$ for ρ and r large†. Then some calculation shows that for

$$e^{i\theta(p,\rho)} = 2^{-2i\rho} e^{-i3\pi/4} e^{-i\rho(\ln\rho - 1)} [p/(\omega(p) - \rho)]^{i\rho} e^{-i\omega(\rho)} \tag{8.18}$$

the limit (8.14) exists and is equal to

$$\phi(x) = 2m^{2s} \sum_{s_3} \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega(k)} (e^{-ikx} a^+(k, s_3)\phi^+(k, s_3) + e^{+ikx} c^-(k, s_3)\phi^-(k, s_3))$$
$$k = (\omega(k), \boldsymbol{k}), \tag{8.19}$$

where the functions $\phi^\pm(k, s_3)$ are exactly the fundamental solutions in the helicity basis, e.g.

$$\phi^+(k, s_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} [(\omega(k) + |\boldsymbol{k}|)/m]^{+s_3} \chi(\boldsymbol{k}, s_3) \\ [(\omega(k) + |\boldsymbol{k}|)/m]^{-s_3} \chi(\boldsymbol{k}, s_3) \end{pmatrix}.$$

To obtain this result was the reason for choosing the Whittaker functions $M_{k,m}(z)$ of the first kind. If we had chosen instead the functions $W_{k,m}(z)$ of the second kind, the limit of, e.g. $\psi^+(z)$, would contain positive and negative frequency parts.

To sum up, we have shown that particles of arbitrary spin s and mass $m = \rho/r \neq 0$ exist in de Sitter space and their fields contract into those of Minkowski space. With the exception of the Klein–Gordon equation (8.1), we had no recourse to field equations. We now return to the investigation of fields of spin $s \leq 1$, defined by means of the field equations derived in § 6. Furthermore, we make the transition to quantum fields. Then the Fourier amplitudes $a^+(\boldsymbol{p}, s_3)$ and $c^-(\boldsymbol{p}, s_3)$ become operators with adjoints

$$c^+(\boldsymbol{p}, s_3) = a^+(\boldsymbol{p}, s_3)^\dagger \qquad a^-(\boldsymbol{p}, s_3) = c^-(\boldsymbol{p}, s_3)^\dagger.$$

As usual $a^\pm(\boldsymbol{p}, s_3)$ and $c^\pm(\boldsymbol{p}, s_3)$ are interpreted as annihilation and creation operators of

† In Mielke (1977) the asymptotic expansion of the Whittaker functions $w_{k,m}(2imz)$ (see Kazarinoff 1955, equation (9.3)) is used, though this formula is applicable only for $|z/2m| > 1$.

particle and anti-particle states, respectively†. They are assumed to obey the commutation or anti-commutation relations

$$\begin{aligned}[][a^{\pm}(\boldsymbol{p}, s_3), c^{\pm}(\boldsymbol{p}', s_3')] \\ &= a^{\pm}(\boldsymbol{p}, s_3)c^{\pm}(\boldsymbol{p}', s_3') - (-1)^{2s}c^{\pm}(\boldsymbol{p}', s_3')a^{\pm}(\boldsymbol{p}, s_3) \\ &= (2\pi)^3\delta_{s_3s_3'}\delta(\boldsymbol{p}-\boldsymbol{p}')\end{aligned} \tag{8.20}$$

in agreement with the connection between spin and statistics. This definition is compatible with the behaviour under scale transformations, e.g.

$$T(d)^{-1}a^{\pm}(\boldsymbol{p}, s_3)T(d) = (\mathrm{e}^{\lambda})^{3/2}a^{\pm}(\mathrm{e}^{\lambda}\boldsymbol{p}, s_3). \tag{8.21}$$

The creation and annihilation operators thus have dimension $-3/2$ for arbitrary spin. Note that to obtain (8.21) we must set the phase factor (8.18) equal to one.

$s = 0$

The solutions (8.9)–(8.10) can be employed for the scalar case if we identify

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \equiv 1,$$

and if the factor $\sqrt{(2)}m^s$ in (8.14)–(8.15) is replaced by one. The commutator function, defined by

$$[\psi(z), \psi(z')^{\dagger}] = \mathrm{i}\Delta(z, z'), \tag{8.22}$$

obviously is a solution of the homogeneous Klein–Gordon equation and obeys

$$\nabla^0(z)\Delta(z, z')|_{z^0=z'^0} = -(z^0)^3\delta^3(z-z') \to -\delta^3(x-x'). \tag{8.23}$$

The limit in (8.23) is obtained upon multiplication with r^{-2} because the contraction has to be carried out with respect to

$$\phi(x) = \lim_{r\to\infty} r^{-1}\psi(z) \tag{8.24}$$

according to the above remark. Furthermore, the distribution $\Delta(z, z')$ has the property

$$\Delta(z, z')|_{z^0=z'^0} = 0 \tag{8.25}$$

so that the scalar field is local.

$s = \frac{1}{2}$

The calculation shows that the solutions (8.9)–(8.10) are solutions of the Dirac equation as well. This is equally valid in the flat case. Note that this property fails to be true if we had used the Whittaker functions of the second kind instead. The determination of the equal-time anti-commutator yields

$$[\psi(z), \psi(z')^{\dagger}]|_{z^0=z'^0} = \frac{1}{2\rho}(z^0)^3\delta^3(z-z') \to \delta^3(x-x'), \tag{8.26}$$

by taking into account the identity

$$\chi(-\boldsymbol{p}, s_3) = (-1)^{s+s_3}\{(p^1 - \mathrm{i}p^2)/\sqrt{[2p(p+p^3)]}\}^{2s_3}\chi(\boldsymbol{p}, -s_3).$$

Hence the Dirac field obeys the principle of Einstein causality.

† Actually, this is a subtle point because the field representation splits into the direct sum of two equivalent representations in contrast to the flat case.

$s=1$

This case is not covered by the above investigations on Weinberg-type fields. The solutions of the Klein–Gordon equation (6.3), or explicitly

$$r^2\Box(z)\psi^0(z)+2z^0\partial^i\psi_i(z)=[2+\Delta(\Delta+3)]\psi^0(z)$$
$$r^2\Box(z)\psi^k(z)-2z^0\partial^k\psi_0(z)=\Delta(\Delta+3)\psi^k(z),$$

and the Lorentz condition (6.7) again can be given the form (8.9). The fundamental solutions $\psi^{\pm}(z^0,\boldsymbol{p};s_3)$ are calculated to be

$$\psi^-(z^0,\boldsymbol{p};\pm1)=N(p,\rho)\psi(\boldsymbol{p},\pm1)wM_{0,\mathrm{i}\rho}(-2\mathrm{i}w)$$
$$\psi^-(z^0,\boldsymbol{p};0)=N(p,\rho)\frac{1}{\sqrt{(\rho^2+\frac{1}{4})}}\left(\frac{\mathrm{i}\boldsymbol{p}}{p}\left(2-w\frac{\mathrm{d}}{\mathrm{d}w}\right)\right)wM_{0,\mathrm{i}\rho}(-2\mathrm{i}w),\tag{8.27}$$

with the basis functions $\psi(\boldsymbol{p},\pm1)$ being defined by

$$\psi^\mu(\boldsymbol{p},\pm1)=\frac{1}{\sqrt{2}}(\mathring{\Lambda}_{\boldsymbol{p}}{}^\mu{}_1\pm\mathrm{i}\mathring{\Lambda}_{\boldsymbol{p}}{}^\mu{}_2)\tag{8.28}$$

and $\mathring{\Lambda}_{\boldsymbol{p}}=\tilde{\pi}(\mathring{A}_{\boldsymbol{p}})$. Furthermore, the positive frequency fundamental solutions are

$$\psi^+(z^0,\boldsymbol{p};s_3)=\psi^-(z^0,\boldsymbol{p};-s_3)^*.\tag{8.29}$$

These solutions are normalised with respect to the scalar product

$$(\psi_1,\psi_2)=-\mathrm{i}\int\Sigma^\mu(z)(\psi_1^+(z)^*_\nu\overleftrightarrow{\nabla}_\mu(z)\psi_2^+(z)^\nu-\psi_1^-(z)^*_\nu\overleftrightarrow{\nabla}_\mu(z)\psi_2^-(z)^\nu).\tag{8.30}$$

Again, the contraction yields the vector field of flat space in the helicity basis, with the limit being taken according to (8.24). There remains the determination of the equal-time commutator, which yields

$$\begin{aligned}[\psi^0(z),\psi^k(z')^\dagger]|_{z^0=z'^0}&\\ &=[\psi^k(z),\psi^0(z')^\dagger]|_{z^0=z'^0}\\ &=-\frac{\mathrm{i}}{\rho^2+\frac{1}{4}}z^0\partial^k((z^0)^3\delta^3(z-z'))\to\frac{\mathrm{i}}{m^2}\frac{\partial}{\partial x_k}\delta^3(x-x'),\end{aligned}\tag{8.31}$$

with all other commutators vanishing. This proves the locality property of the vector field.

9. Commutator function

We now want to calculate the explicit invariant form of the commutator function. For later purposes we do this for arbitrary dimension d of de Sitter space–time.

To begin with, we determine the solutions of the equation

$$[z^0z_0\partial^\mu\partial_\mu-(d-2)z^0\partial_0]\psi(z)=\Delta(\Delta+d-1)\psi(z)\tag{9.1}$$

with $\mu = 0, 1, \ldots, d-1$ and $\Delta = -\frac{1}{2}(d-1) + \mathrm{i}\rho$. They read

$$\psi(z) = \int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}} [\mathrm{e}^{+\mathrm{i}\boldsymbol{p}\boldsymbol{z}} \bar{N}^*(p,\rho)(pz^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz^0)^* a^+(\boldsymbol{p})$$
$$+ \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\boldsymbol{z}} \bar{N}(p,\rho)(pz^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz^0) c^-(\boldsymbol{p})] \tag{9.2}$$

where for convenience we have chosen Hankel functions. The modified normalisation constant is

$$\bar{N}(p,\rho) = \tfrac{1}{2}\sqrt{(\pi)}\, \mathrm{e}^{-\pi\rho/2} (p)^{-\frac{1}{2}(d-1)}, \tag{9.3}$$

with the phase factor being omitted. By making use of the commutation relations

$$[a^\pm(\boldsymbol{p}), c^\pm(\boldsymbol{p}')] = (2\pi)^{d-1}\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{9.4}$$

the commutator functions

$$[\psi^\pm(z), \psi^\pm(z')^\dagger] = \mathrm{i}\Delta^\pm(z,z') \tag{9.5}$$

are obtained to be

$$\Delta^-(z_1,z_2) = -\Delta^+(z_1,z_2)$$
$$= \mathrm{i}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^{d-1}} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\boldsymbol{z}_1-\boldsymbol{z}_2)} |\bar{N}(p,\rho)|^2 (pz_1^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz_1^0)(pz_2^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz_2^0)^*. \tag{9.6}$$

The integration over the angular variables can be performed with the result

$$\Delta^-(z_1,z_2) = \mathrm{i}(2\pi)^{-\frac{1}{2}(d-1)}|\boldsymbol{z}_{12}|^{-\frac{1}{2}(d-3)} \int_0^\infty \mathrm{d}p\, p^{\frac{1}{2}(d-1)} J_{\frac{1}{2}(d-3)}(p|\boldsymbol{z}_{12}|)$$
$$\times |\bar{N}(p,\rho)|^2 (pz_1^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz_1^0)(pz_2^0)^{\frac{1}{2}(d-1)} H^{(1)}_{\mathrm{i}\rho}(pz_2^0)^*, \tag{9.7}$$

and where $\boldsymbol{z}_{12} = \boldsymbol{z}_1 - \boldsymbol{z}_2$. To make the integral absolutely convergent we analytically continue z into the backward tube,

$$z \to (z^0 + \mathrm{i}\epsilon, \boldsymbol{z}), \tag{9.8}$$

which is a reasonable procedure due to the fact that the conformal group acts as a transitive transformation group on this domain (Rühl 1972, 1973a,b, Grensing 1976). If in addition the Hankel functions are replaced by modified Bessel functions, we find

$$\Delta^-(z_1,z_2) = \mathrm{i}\pi^{-1}|\boldsymbol{z}_{12}|^{-\frac{1}{2}(d-3)}(z_1^0 z_2^0)^{\frac{1}{2}(d-1)}$$
$$\times (2\pi)^{-\frac{1}{2}(d-1)} \int_0^\infty \mathrm{d}p\, p^{\frac{1}{2}(d-1)} K_{\mathrm{i}\rho}(-\mathrm{i}p(z_1^0 + \mathrm{i}\epsilon))$$
$$\times K_{\mathrm{i}\rho}(+\mathrm{i}p(z_2^0 - \mathrm{i}\epsilon)) J_{\frac{1}{2}(d-3)}(p|\boldsymbol{z}_{12}|). \tag{9.9}$$

This integral is evaluated by means of (Magnus *et al* 1966, p 103)

$$\int_0^\infty \mathrm{d}x\, x^{\nu+1} K_\mu(ax) K_\mu(bx) J_\nu(cx)$$
$$= \tfrac{1}{2}\sqrt{(\tfrac{1}{2}\pi)}(ab)^{-\nu-1} c^\nu \Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)(\omega^2-1)^{-\frac{1}{2}\nu-\frac{1}{4}} P^{-\nu-\frac{1}{2}}_{\mu+\frac{1}{2}}(\omega), \tag{9.10}$$

where $\omega = \frac{1}{2}(a^2+b^2+c^2)/bc$ and $\mathrm{Re}(\nu \pm \mu) > -1$, $\mathrm{Re}\,\nu > -1$, $\mathrm{Re}\,a > 0$, $\mathrm{Re}\,b > 0$, $c > 0$.

We then obtain

$$\Delta^-(z_1, z_2)=\tfrac{1}{2}\mathrm{i}(2\pi)^{-\frac{1}{2}d}\Gamma(\tfrac{1}{2}(d-1)+\mathrm{i}\rho)\Gamma(\tfrac{1}{2}(d-1)-\mathrm{i}\rho)$$
$$\times[(p_{12})^2-1]^{-\frac{1}{4}(d-2)}P^{-\frac{1}{2}(d-2)}_{-\frac{1}{2}+\mathrm{i}\rho}(-p_{12}(\epsilon)), \tag{9.11}$$

where the geodesic distance

$$p_{12}=1+\frac{(z_1-z_2)^2}{2z_1^0z_2^0}=p(z_1;z_2) \tag{9.12}$$

of the points z_1 and z_2 has been introduced (cf Gel'fand *et al* 1966) and $p_{12}(\epsilon)=p(z_1^0+\mathrm{i}\epsilon, \mathbf{z}_1;\ z_2^0-\mathrm{i}\epsilon, \mathbf{z}_2)$. This result can be further simplified with the aid of the following relation between the Legendre function and the hypergeometric function (Magnus *et al* 1966, p 52)

$${}_2F_1(a, b; \tfrac{1}{2}(a+b+1); z)=\Gamma(\tfrac{1}{2}(a+b+1))[z(z-1)]^{\frac{1}{4}(1-a-b)}P^{\frac{1}{2}(1-a-b)}_{\frac{1}{2}(a-b-1)}(1-2z) \tag{9.13}$$
$$|\arg z|<\pi, \qquad |\arg(z-1)|<\pi, \qquad z\notin[0, 1].$$

Then (9.11) takes the closed form

$$\Delta^-(z_1, z_2)=-\mathrm{i}(4\pi)^{-\frac{1}{2}d}\frac{\Gamma(\frac{1}{2}(d-1)+\mathrm{i}\rho)\Gamma(\frac{1}{2}(d-1)-\mathrm{i}\rho)}{\Gamma(\frac{1}{2}d)}$$
$$\times{}_2F_1(\tfrac{1}{2}(d-1)+\mathrm{i}\rho, \tfrac{1}{2}(d-1)-\mathrm{i}\rho; \tfrac{1}{2}d; \tfrac{1}{2}(1+p_{12}(\epsilon))). \tag{9.14}$$

The hypergeometric function has a cut from $1\leqslant z<\infty$ so that the ϵ-prescription must be taken into account for time-like or light-like separated points only. Furthermore, because of

$$\tfrac{1}{2}(1+p_{12}(\epsilon))=\tfrac{1}{2}(1+p_{12})\left(1+\mathrm{i}\epsilon\frac{z_1^0-z_2^0}{z_1^0z_2^0}\right) \tag{9.15}$$

the distribution (9.14) turns out to be explicitly invariant (cf § 4).

Finally, starting from (9.11), the commutator function $\Delta(z_1, z_2)=\Delta^+(z_1, z_2)+\Delta^-(z_1, z_2)$ for physical space–time dimension $d=4$ is determined to be

$$\Delta(z_1, z_2)=\frac{1}{4\pi}\epsilon(z_1^0, z_2^0)\frac{\mathrm{d}}{\mathrm{d}p_{12}}(\theta(p_{12}-1)P_{-\frac{1}{2}+\mathrm{i}\rho}(p_{12})) \tag{9.16}$$

with

$$\epsilon(z_1^0, z_2^0)=\begin{cases}+1 & \dfrac{z_1^0-z_2^0}{z_1^0z_2^0}>0\\[2ex] -1 & \dfrac{z_1^0-z_2^0}{z_1^0z_2^0}<0,\end{cases} \tag{9.17}$$

which agrees up to factor r^{-2} with the result obtained in Tagirov (1973).

10. Irreducibility of the field representations and some comments on mass zero particles

Our discussion of free fields in de Sitter space is not yet complete because it remains to prove that the fields transform irreducibly. A rigorous proof would require a detailed analysis of time inversion invariance. However, this is a difficult problem which has not been solved so far. To circumvent it, we will henceforth assume that the solutions of the field equations have definite parity with respect to time inversion, without specifying their explicit form. These solutions constitute an invariant sub-space, which we expect to transform irreducibly under elements of the identity component of the de Sitter group. The following arguments show that this suggestion is reasonable (Streitz 1976).

For simplicity we only discuss the scalar case. To begin with, we introduce functions $f(\xi)$ defined on $\xi^2<0$. Obviously, the quasi-regular representation

$$T(R)f(\xi)=f(R^{-1}\xi) \tag{10.1}$$

with $R\in SO_0(1,4)$ is not irreducible. To achieve this, we require f to be an harmonic, homogeneous function of degree $\Delta=-\frac{3}{2}+i\rho$,

$$\Box(\xi)f(\xi)=0 \tag{10.2}$$

$$f(\alpha\xi)=\alpha^{\Delta}f(\xi), \tag{10.3}$$

where $\Box(\xi)=(\partial/\partial\xi^a)(\partial/\partial\xi_a)$ and $\alpha>0$. Furthermore if f is assumed to be an even or odd function, the sub-space so defined will be irreducible. This property can be proved rigorously by means of the Gel'fand–Graev transformation (Gel'fand *et al* 1966) for functions f which are defined instead on $\xi^2>0$, $\xi_0>0$ and obey (10.2)–(10.3) (Vilenkin 1968). However, for $\xi^2<0$ this transformation is known for $d=3$ only, so that a proof of our assertion is not available. Taking for granted this property, we now show how the Klein–Gordon equation appears in this context. To this end we make use of (10.3) and define

$$\psi(z)=r^{-\Delta}f(\xi), \tag{10.4}$$

with ξ and z being related according to (2.18) and $r=\surd(-\xi^2)$. In terms of the variables z and r the operator $\Box(\xi)$ reads

$$\Box(\xi)=\frac{1}{r^4}\frac{\partial}{\partial r}r^4\frac{\partial}{\partial r}+\Box(z), \tag{10.5}$$

and we obtain

$$\Box(\xi)f(\xi)=0=r^{\Delta-2}[r^2\Box(z)-\Delta(\Delta+3)]\psi(z). \tag{10.6}$$

Thus, the homogeneity property can be used to restrict the functions f on $\xi^2<0$ to $\xi^2=-r^2$, which together with the requirement that f is harmonic yields the Klein–Gordon equation.

We add some remarks on mass zero particles in de Sitter space. At first we need information about the irreducible sub-representations occurring in the reduction of the quasi-regular representation (10.1) on functions over $\xi^2=-r^2$. A thorough investigation, which we omit for the sake of brevity, shows that two types of representations appear. The first is contained in the continuous principal series and carries the index $(0;\frac{1}{2}(d-1)+i\rho)$ with $\rho\neq0$. They have been associated above with scalar particles of non-zero mass for $d=4$. The second type is contained in the exceptional series with index $(l;-1)$, where l is a multi-index. Only for $d=2$ do these representations belong to the discrete series, and for $d=3$ to the principal series with $\rho=0$. We expect the representations to be associated with particles of mass zero. To confirm this supposition, we note that for $\Delta=-1$ and spin $l=0$ the field (cf Börner and Dürr 1969)

$$\phi(z)=\frac{1}{z^0}\psi(z) \tag{10.7}$$

has canonical dimension $\delta=1$,

$$T(d)\phi(z)=(\mathrm{e}^{\lambda})^{-1}\phi(\mathrm{e}^{-\lambda}z). \tag{10.8}$$

Furthermore, it obeys the equation

$$\partial_\mu\partial^\mu\phi(z)=0 \tag{10.9}$$

so that the de Sitter group may be enlarged to the conformal group. A more sensitive test is obtained for $l=1$, where for the massless case some kind of gauge invariance should occur. We thus determine those values of Δ for which the field equations are invariant under the substitution

$$\psi^\mu(z)\rightarrow\psi'^\mu(z)=\psi^\mu(z)+\nabla^\mu(z)\Lambda(z). \tag{10.10}$$

Since ψ'^μ must be a solution of the Klein–Gordon equation together with ψ^μ, the arbitrary function Λ must obey the condition

$$T(C_{\mathrm{II}})\nabla^\mu(z)\Lambda(z)=[2+\Delta(\Delta+3)]\nabla^\mu(z)\Lambda(z)=\nabla^\mu(z)\Box(z)\Lambda(z), \tag{10.11}$$

and the invariance of the Lorentz condition requires

$$\Box(z)\Lambda(z)=0, \tag{10.12}$$

which yields $\Delta=-1$. We have not investigated in detail the solutions of the field equations for mass zero. However, in the scalar case they are easily obtained and can be shown to contract into the conformal invariant solutions of the flat case. Furthermore, the field turns out to be local (cf also Tagirov 1973), in contrast to a result given in Börner and Dürr (1969).

11. The renormalisation problem for the model of a scalar particle in the de Sitter background field

In the preceding sections we have given a rather detailed investigation of free field theory in de Sitter space. We now want to treat the simplest case of an interaction problem, that of a scalar quantum field interacting with its classical background geometry. The metric tensor $g_{\mu\nu}(z)$ thus plays the role of an external field. If only the first quantum correction is taken into account, the theory is fully determined by the effective Lagrangian $\mathscr{L}_{\mathrm{M}}^{(1)}$ of the matter field. Then the total Lagrangian $\mathscr{L}=\mathscr{L}_{\mathrm{G}}+\mathscr{L}_{\mathrm{M}}^{(1)}$, with $\mathscr{L}_{\mathrm{G}}$ the gravitational Lagrangian, generates the irreducible vertex functions with only one closed loop. It is a known fact that $\mathscr{L}_{\mathrm{M}}^{(1)}$ is related to the Feynman propagator $\Delta_{\mathrm{F}}(z_1,z_2)$ of the matter field in the presence of the background by the equation (for a derivation see, e.g. Brown and Duff 1975):

$$\frac{\partial}{\partial m^2}\mathscr{L}_{\mathrm{M}}^{(1)}(z)=\frac{\mathrm{i}}{2}\Delta_{\mathrm{F}}(z,z). \tag{11.1}$$

The Feynman propagator is easily obtained from the results of § 8 and will be given below. Hence the effective Langrangian is known explicitly.

Does this theory obey the requirement of being renormalisable? This question has been answered in the affirmative by Candelas and Raine (1975), and Dowker and Critchley (1976) by applying the dimensional regularisation technique directly in coordinate space. We present a corrected proof of renormalisability, which rigorously takes into account the pole proportional to the square of the curvature. Moreover, a slight modification results from our definition $m=\rho/r$ of the mass.

First of all, by means of equation (9.14) the Feynman propagator

$$\Delta_F(z_1, z_2)=\begin{cases} +r^{-d+2}\Delta^+(z_1, z_2) & \dfrac{z_1^0-z_2^0}{z_1^0z_2^0}>0 \\ -r^{-d+2}\Delta^-(z_1, z_2) & \dfrac{z_1^0-z_2^0}{z_1^0z_2^0}<0 \end{cases} \tag{11.2}$$

is determined to be

$$\Delta_F(z_1, z_2)=-\mathrm{i}r^2(4\pi r^2)^{-\frac{1}{2}d}\frac{\Gamma(\frac{1}{2}(d-1)+\mathrm{i}mr)\Gamma(\frac{1}{2}(d-1)-\mathrm{i}mr)}{\Gamma(\frac{1}{2}d)}$$
$$\times {}_2F_1(\tfrac{1}{2}(d-1)+\mathrm{i}mr, \tfrac{1}{2}(d-1)-\mathrm{i}mr; \tfrac{1}{2}d; \tfrac{1}{2}(1+p_{12})-\mathrm{i}\epsilon) \tag{11.3}$$

which agrees up to a minus sign with the result obtained by Candelas and Raine (1975) and Dowker and Critchley (1976) by means of the Schwinger–De Witt proper-time technique (De Witt 1964, 1975). In particular, we have shown that the Fulling phenomenon (Fulling 1973) does not occur (cf, however, Candelas and Raine 1975). Note that the commutator functions $\Delta^{\pm}$ have been multiplied by a factor r^{-d+2} to assign to Δ_F the correct dimension.

We define $d=2\omega$ and analytically continue to complex values of ω. The coincidence limit of (11.3) exists for $\operatorname{Re}\omega<1$ and reads

$$\Delta_F(z, z)=-\mathrm{i}r^2(4\pi r^2)^{-\omega}\Gamma(1-\omega)\frac{\Gamma(\omega-\frac{1}{2}+\mathrm{i}mr)\Gamma(\omega-\frac{1}{2}-\mathrm{i}mr)}{\Gamma(\frac{1}{2}+\mathrm{i}mr)\Gamma(\frac{1}{2}-\mathrm{i}mr)}. \tag{11.4}$$

This is an analytic function of ω except for simple poles with the following Laurent series about $\omega=2$,

$$\Delta_F(z, z)=-\frac{\mathrm{i}}{16\pi^2r^2}(m^2r^2-2)[(\omega-2)^{-1}+\psi(\tfrac{3}{2}+\mathrm{i}mr)+\psi(\tfrac{3}{2}-\mathrm{i}mr)$$
$$-\ln 4\pi r^2+C-1]+\mathrm{O}(\omega-2), \tag{11.5}$$

where now $\psi(z)=\Gamma'(z)/\Gamma(z)$ and C is Euler's constant. The Laurent series of the effective Lagrangian is then found to be

$$\mathscr{L}_M^{(1)}=\frac{1}{32\pi^2r^2}\bigg((\tfrac{1}{2}m^4r^2+\tfrac{1}{4}m^2)(\omega-2)^{-1}+\int \mathrm{d}m^2(m^2r^2+\tfrac{1}{4})(\psi(\tfrac{3}{2}+\mathrm{i}mr)+\psi(\tfrac{3}{2}-\mathrm{i}mr))$$
$$+(\tfrac{1}{2}m^4r^2+\tfrac{1}{4}m^2)(-\ln 4\pi r^2+C-1)\bigg)+\mathrm{O}(\omega-2), \tag{11.6}$$

which displays the form of the singularity at $\omega=2$. The effective Lagrangian thus has poles proportional to r^0 and r^{-2}. However, as is known from standard theory (De Witt

1964, 1975), there should exist a further pole proportional to r^{-4}. It has been lost by the illegitimate interchange of the integration and expansion process.

The existence of the r^{-4} pole can be demonstrated as follows. We asymptotically expand the effective Lagrangian about $r=\infty$, which essentially amounts to a perturbation expansion in powers of the curvature. This we achieve by means of (Magnus *et al* 1966, p 13)

$$\Gamma(\alpha+\mathrm{i}x)\Gamma(\alpha-\mathrm{i}x)/\Gamma(\beta+\mathrm{i}x)\Gamma(\beta-\mathrm{i}x)\sim(x^2)^{\alpha-\beta}(a_0+a_1x^{-2}+a_2x^{-4}+\mathrm{O}(x^{-6}))$$

with $a_0=1$, $a_1=\frac{1}{3}(B_3(\alpha)-B_3(\beta))$, $a_2=\frac{1}{18}(B_3(\alpha)-B_3(\beta))^2-\frac{1}{10}(B_5(\alpha)-B_5(\beta))$ and $B_n(z)$ the Bernoulli polynomials. Hence we find for the propagator

$$\Delta_{\mathrm{F}}(z,z)\sim-\frac{\mathrm{i}}{(4\pi)^{\omega}}\sum_{k=0}^{\infty}(r^2)^{-k}(m^2)^{\omega-k-1}J_k(\omega) \tag{11.7}$$

with

$$\begin{aligned}J_0(\omega)&=\Gamma(1-\omega)\\ J_1(\omega)&=-\tfrac{1}{3}(\omega-\tfrac{1}{2})(\omega-\tfrac{3}{2})\Gamma(2-\omega)\\ J_2(\omega)&=\tfrac{1}{360}(\omega-\tfrac{1}{2})(\omega-\tfrac{3}{2})(20\omega^2-56\omega+15)\Gamma(3-\omega)\\ &\ \ \vdots\end{aligned} \tag{11.8}$$

which shows that the $J_k(\omega)$ are regular for $k\geqslant 2$. Consequently, the asymptotic expansion of the effective Lagrangian,

$$\mathscr{L}_{\mathrm{M}}^{(1)}\sim\frac{1}{2(4\pi)^{\omega}}\sum_{k=0}^{\infty}(r^2)^{-k}(m^2)^{\omega-k}\frac{J_k(\omega)}{\omega-k}, \tag{11.9}$$

in fact has poles proportional to r^0, r^{-2}, r^{-4}. The Laurent series for the coefficients of the expansion are

$$\begin{aligned}\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_0(\omega)}{\omega}&=\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{2}\frac{1}{2}\left[(\omega-2)^{-1}+\ln\left(\frac{m^2}{4\pi}\right)+C-\frac{3}{2}\right]+\mathrm{O}(\omega-2)\\ \frac{1}{2m^2}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_1(\omega)}{\omega-1}&=\frac{1}{2m^2}\left(\frac{m^2}{4\pi}\right)^{2}\frac{1}{4}\left[(\omega-2)^{-1}+\ln\left(\frac{m^2}{4\pi}\right)+C+\frac{5}{3}\right]+\mathrm{O}(\omega-2)\\ \frac{1}{2m^4}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_2(\omega)}{\omega-2}&=-\frac{1}{2m^4}\left(\frac{m^2}{4\pi}\right)^{2}\frac{17}{480}\left[(\omega-2)^{-1}+\ln\left(\frac{m^2}{4\pi}\right)+C+\frac{64}{51}\right]+\mathrm{O}(\omega-2).\end{aligned} \tag{11.10}$$

We take into account the r^{-4} pole by making use of the freedom in the choice of an integration constant,

$$\mathscr{L}_{\mathrm{M}}^{(1)}(z)=\tfrac{1}{2}\mathrm{i}\int \mathrm{d}m^2\Delta_{\mathrm{F}}(z,z)+r^{-4}C(\omega) \tag{11.11}$$

where $C(\omega)$ is independent of m. It is tempting to choose the integration constant to be equal to the term $k=2$ of (11.9), as has been done by Dowker and Critchley (1976). However, the Laurent series of $\mathscr{L}_{\mathrm{M}}^{(1)}$ contains terms proportional to r^{-4}. We extract them by asymptotically expanding (11.6). Using (Norlund 1924, p 101)

$$\psi(\tfrac{3}{2}+\mathrm{i}mr)+\psi(\tfrac{3}{2}-\mathrm{i}mr)\sim\ln(mr)^2+\tfrac{11}{12}(mr)^{-2}-\tfrac{127}{480}(mr)^{-4}+\mathrm{O}(r^{-6}),$$

we obtain

$$\begin{aligned}\mathscr{L}^{(1)}_{\mathrm{M}}\sim\frac{1}{32\pi^2}\bigg\{&\left(\frac{1}{2}m^4+\frac{1}{4}m^2r^{-2}\right)(\omega-2)^{-1}+\frac{1}{2}m^4\left[\ln\left(\frac{m^2}{4\pi}\right)+C-\frac{3}{2}\right]\\&+\frac{1}{4}m^2r^{-2}\left[\ln\left(\frac{m^2}{4\pi}\right)+C+\frac{5}{3}\right]-\frac{17}{480}r^{-4}\ln(mr)^2+\mathrm{O}(r^{-6})\bigg\}+\mathrm{O}(\omega-2)\end{aligned}\tag{11.12}$$

which proves the assertion. We can determine the integration constant by comparing (11.12) with the Laurent series of (11.9),

$$\begin{aligned}\mathscr{L}^{(1)}_{\mathrm{M}}\sim\frac{1}{32\pi^2}\bigg\{&\left(\frac{1}{2}m^4+\frac{1}{2}m^2r^{-2}-\frac{17}{480}r^{-4}\right)(\omega-2)^{-1}+\frac{1}{2}m^4\left[\ln\left(\frac{m^2}{4\pi}\right)+C-\frac{3}{2}\right]\\&+\frac{1}{4}m^2r^{-2}\left[\ln\left(\frac{m^2}{4\pi}\right)+C+\frac{5}{3}\right]-\frac{17}{480}r^{-4}\left[\ln\left(\frac{m^2}{4\pi}\right)+C+\frac{64}{51}\right]+\mathrm{O}(\omega-2)\bigg\}\\&+\frac{1}{2(4\pi)^{\omega}}\sum_{k=3}^{\infty}(r^2)^{-k}(m^2)^{\omega-k}\frac{J_k(\omega)}{\omega-k},\end{aligned}\tag{11.13}$$

so that $C(\omega)$ must be equal to

$$C(\omega)=\frac{1}{2m^4}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_2(\omega)}{\omega-2}+\frac{1}{2}r^4\left(\frac{1}{4\pi r^2}\right)^{\omega}\frac{17}{480}\ln(mr)^2.\tag{11.14}$$

The infinities of the effective Lagrangian can be absorbed by a re-definition of the gravitational Lagrangian

$$\mathscr{L}_{\mathrm{G}}=(16\pi G)^{-1}R-2\lambda+(\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}+\gamma R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}),\tag{11.15}$$

where G is the gravitational and λ the cosmological constant. The last term on the right-hand side of (11.15) has been added according to the general renormalisation procedure (De Witt 1964, 1975, Isham *et al* 1975). We define renormalised constants by

$$\begin{aligned}-2\lambda_{\mathrm{R}}&=-2\lambda+\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_0(\omega)}{\omega}-2\omega(2\omega-1)(16\pi G_{\mathrm{R}})^{-1}\\&=-2\omega(2\omega-1)(16\pi G)^{-1}+\frac{1}{2m^2}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_1(\omega)}{\omega-1}\end{aligned}\tag{11.16}$$

$$\begin{aligned}&4\omega^2(2\omega-1)^2\left(\alpha_{\mathrm{R}}+\frac{1}{2\omega}\beta_{\mathrm{R}}+\frac{1}{\omega(2\omega-1)}\gamma_{\mathrm{R}}\right)\\&\qquad=4\omega^2(2\omega-1)^2\left(\alpha+\frac{1}{2\omega}\beta+\frac{1}{\omega(2\omega-1)}\gamma\right)+\frac{1}{2m^4}\left(\frac{m^2}{4\pi}\right)^{\omega}\frac{J_2(\omega)}{\omega-2},\end{aligned}$$

and after all obtain a finite total Lagrangian at $\omega=2$,

$$
\begin{aligned}
\mathscr{L}= & -\frac{3}{4\pi G_{\mathrm{R}} r^{2}}-2\lambda_{\mathrm{R}}+\frac{144}{r^{4}}\left(\alpha_{\mathrm{R}}+\frac{1}{4}\beta_{\mathrm{R}}+\frac{1}{6}\gamma_{\mathrm{R}}\right)+\frac{1}{32\pi^{2}}\left[\frac{1}{4}m^{4}-\frac{2}{3}m^{2}r^{-2}\right. \\
& -\left(\frac{1}{2}m^{4}+\frac{1}{4}m^{2}r^{-2}-\frac{17}{480}r^{-4}\right)\ln(m^{2}r^{2})+r^{-2}\int \mathrm{d}m^{2}\left(m^{2}r^{2}+\frac{1}{4}\right) \\
& \left.\times\left(\psi\left(\frac{3}{2}+imr\right)+\psi\left(\frac{3}{2}-imr\right)\right)\right]. \qquad (11.17)
\end{aligned}
$$

Thus, the quantum theory of a scalar field in the de Sitter background proves to be renormalisable. However, this result should not be taken too seriously in view of the recent research on the renormalisation problem of quantum gravity (Isham *et al* 1975) by means of the methods of non-Abelian gauge theories.

Appendix 1. Representations of the continuous principal series

We give a brief account of those representations of G which originate from the Bruhat decomposition (2.11).

In actual computations it is convenient to use instead of G an isomorphic group $\hat{G}$ defined by

$$\hat{g}=\phi^{-1}g\phi, \qquad \phi=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}1\\ \mathrm{i}1 & 1\end{pmatrix} \qquad (\mathrm{A}.1)$$

so that the nilpotent factors

$$N^{-}=\{g(\boldsymbol{a})\}_{\boldsymbol{a}\in\mathbb{R}^{3}} \qquad N^{+}=\{g(\boldsymbol{c})\}_{\boldsymbol{c}\in\mathbb{R}^{3}}$$

take an explicit nilpotent form. Then the derivation of the representations $\chi=(l;-\frac{3}{2}+\mathrm{i}\rho)$ of the continuous principal series proceeds along well known lines (see Streitz 1976); a similar investigation for the spin covering of the Euclidean conformal group $SO_0(1,5)$ is carried out in Grensing (1975b). They are induced by the unitary and irreducible representations of the subgroup G' with elements $g'=g(\mathring{A})g(d)g(\boldsymbol{c})$, which are trivial on N^{+}, and read

$$
\begin{aligned}
& U^{\chi}(\boldsymbol{a})f(\boldsymbol{x})=f(\boldsymbol{x}-\boldsymbol{a}) \\
& U^{\chi}(\mathring{A})f(\boldsymbol{x})=D^{(l)}(\mathring{A})f(\Lambda^{-1}\boldsymbol{x}) \\
& U^{\chi}(d)f(\boldsymbol{x})=(\mathrm{e}^{\lambda})^{-\frac{3}{2}+\mathrm{i}\rho}f(\mathrm{e}^{-\lambda}\boldsymbol{x}) \qquad (\mathrm{A}.2)\\
& U^{\chi}(\boldsymbol{c})f(\boldsymbol{x})=(1+2\boldsymbol{x}\cdot\boldsymbol{c}+\boldsymbol{x}^{2}\boldsymbol{c}^{2})^{-\frac{3}{2}+\mathrm{i}\rho}D^{(l)}(\mathring{A}(\boldsymbol{x},\boldsymbol{c}))f(\boldsymbol{x}+\boldsymbol{x}^{2}\boldsymbol{c}/1+2\boldsymbol{x}\cdot\boldsymbol{c}+\boldsymbol{x}^{2}\boldsymbol{c}^{2})
\end{aligned}
$$

with

$$\mathring{A}(\boldsymbol{x},\boldsymbol{c})=1+\hat{\boldsymbol{x}}\,\hat{\boldsymbol{c}}/(1+2\boldsymbol{x}\cdot\boldsymbol{c}+\boldsymbol{x}^{2}\boldsymbol{c}^{2})^{1/2}. \qquad (\mathrm{A}.3)$$

These representations are unitary with respect to the scalar product

$$(f_1,f_2)=\int \mathrm{d}\boldsymbol{x}\, f_1^{\dagger}(\boldsymbol{x})f_2(\boldsymbol{x}). \qquad (\mathrm{A}.4)$$

The computation of the infinitesimal operators

$$\bar{P}_i = \mathrm{i}\frac{\partial}{\partial a^i} g(\boldsymbol{a})|_{a=0}, \qquad \bar{M}_{ik} = \mathrm{i}\frac{\partial}{\partial \alpha^{ik}} g(\mathring{A})|_{\alpha=0},$$
$$\bar{D} = -\mathrm{i}\frac{\partial}{\partial \lambda} g(d)|_{\lambda=0}, \qquad \bar{K}_i = \mathrm{i}\frac{\partial}{\partial c^i} g(\boldsymbol{c})|_{c=0}, \tag{A.5}$$

being related to the operators M_{ab} by

$$\bar{P}_i = M_{i5} - M_{i0}, \qquad \bar{M}_{ik} = M_{ik}, \qquad \bar{D} = M_{50}, \qquad \bar{K}_i = M_{i5} + M_{i0}, \tag{A.6}$$

yields

$$\begin{aligned} &U^{\chi}(\bar{P}_i) = \mathrm{i}\partial_i \\ &U^{\chi}(\bar{M}_{ik}) = D^{(l)}(M_{ik}) + \mathrm{i}(x_i\partial_k - x_k\partial_i) \\ &U^{\chi}(\bar{D}) = \mathrm{i}(-\Delta + x^i\partial_i) \\ &U^{\chi}(\bar{K}_i) = 2(\mathrm{i}\Delta g_{ik} - D^{(l)}(M_{ik}))x^k - \mathrm{i}(\boldsymbol{x}^2 g_{ik} + 2x_ix_k)\partial^k. \end{aligned} \tag{A.7}$$

In terms of the basis (A.5) of the Lie algebra the Casimir operators read

$$C_{\mathrm{II}} = \tfrac{1}{2}\bar{M}_{ik}\bar{M}^{ik} - \bar{K}_i\bar{P}^i - \bar{D}^2 + 3\mathrm{i}\bar{D} \tag{A.8}$$

$$\begin{aligned} C_{\mathrm{IV}} = &-\tfrac{1}{2}\bar{M}_{ik}\bar{M}^{ik}\bar{K}_j\bar{P}^j - \tfrac{1}{2}\bar{M}_{jk}\bar{M}^{ki}\bar{K}_i\bar{P}^j - \tfrac{1}{2}\bar{M}_{jk}\bar{M}^{ki}\bar{K}^j\bar{P}_i \\ &-\tfrac{1}{2}\bar{D}\bar{D}\bar{M}_{ij}\bar{M}^{ij} - \bar{D}\bar{M}_{ij}\bar{K}^i\bar{P}^j - \tfrac{1}{4}\bar{K}_i\bar{K}^i\bar{P}_j\bar{P}^j + \tfrac{1}{4}\bar{K}^i\bar{K}^j\bar{P}_i\bar{P}_j \\ &+3\mathrm{i}\tfrac{1}{2}\bar{M}_{ij}\bar{M}^{ij}\bar{D} + 3\mathrm{i}\tfrac{1}{2}\bar{M}^{ji}\bar{K}_j\bar{P}_i - \mathrm{i}\bar{D}\bar{K}_i\bar{P}^i + \bar{M}_{ij}\bar{M}^{ij} - 2\bar{K}_i\bar{P}^i, \end{aligned} \tag{A.9}$$

which may be used to calculate the eigenvalues (2.28) of $U^{\chi}(C_{\mathrm{II}})$ and $U^{\chi}(C_{\mathrm{IV}})$. For the convenience of the reader we also quote the second Casimir operator in case of the field representation,

$$\begin{aligned} T(C_{\mathrm{IV}}) = &\mathrm{i}\Sigma_{ik}\Sigma^{ji}\Sigma^{k}{}_{j} + \tfrac{3}{2}\Sigma_{ik}\Sigma^{ik} \\ &+2\mathrm{i}\Sigma_{ik}\Sigma^{ji}\Sigma^{k\mu}z_\mu\partial_j + \mathrm{i}\Sigma_{ik}\Sigma^{ik}\Sigma^{j\mu}z_\mu\partial_j + 4\Sigma_{ik}\Sigma^{ji}z^k\partial_j \\ &+2\Sigma_{ik}\Sigma^{ik}z^j\partial_j - 2\Sigma_{ik}\Sigma^{k\mu}z_\mu\partial^i - \Sigma_{ik}\Sigma^{ik}z^\mu\partial_\mu - 2\mathrm{i}\Sigma^{\mu k}z_\mu\partial_k \\ &-2\Sigma_{ik}\Sigma^{i\mu}z_\mu z^k\partial^j\partial_j + 2\Sigma_{ik}\Sigma^{i\mu}z_\mu z^j\partial_j\partial^k + 2\Sigma_{ik}\Sigma^{j\mu}z_\mu z^k\partial^i\partial_j \\ &-\Sigma_{\mu k}\Sigma^{k\nu}z^\mu z_\nu\partial_i\partial^i - \Sigma_{k\mu}\Sigma^{i\nu}z^\mu z_\nu\partial_i\partial^k + 2\Sigma_{ki}\Sigma^{i\mu}z_\mu z^\nu\partial_\nu\partial^k \\ &+\tfrac{1}{2}\Sigma_{ik}\Sigma^{ik}z^\mu z^\nu\partial_\mu\partial_\nu - 2\Sigma_{ik}\Sigma^{ji}z^k z^\mu\partial_\mu\partial_j - \Sigma_{ik}\Sigma^{ik}z^\mu z^j\partial_\mu\partial_j \\ &+\Sigma_{ik}\Sigma^{ji}z_\mu z^\mu\partial^k\partial_j + \tfrac{1}{2}\Sigma_{ik}\Sigma^{ik}z_\mu z^\mu\partial_j\partial^j - \mathrm{i}\Sigma^{k\mu}z_k z_\mu\partial^j\partial_j \\ &+\mathrm{i}\Sigma^{k\mu}z_\mu z^j\partial_j\partial_k - 2\mathrm{i}\Sigma^{k\mu}z_\mu z^\nu\partial_\nu\partial_k, \end{aligned} \tag{A.10}$$

which is needed in § 6.

Appendix 2. Discrete transformations

The full de Sitter group has four connected components. As representatives of the decomposition of SO(1, 4) with respect to the identity component $\mathrm{SO}_0(1,4)$,

$$\mathrm{SO}(1,4) = \sum_J \mathrm{SO}_0(1,4)R_J \qquad J = \mathrm{I, P, T, PT}, \tag{A.11}$$

we choose $R_{\mathrm{I}} = \mathbb{1}$,

$$R_{\mathrm{P}} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}, \qquad R_{\mathrm{T}} = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ & & & & -1 \end{pmatrix}, \tag{A.12}$$

and $R_{\mathrm{PT}} = R_{\mathrm{P}} R_{\mathrm{T}}$. They act on z in the canonical way,

$$R_J \,.\, z = \Lambda_J z. \tag{A.13}$$

In addition we introduce for later purposes

$$R'_J = \begin{pmatrix} \Lambda_J & \\ & 1 \end{pmatrix}, \tag{A.14}$$

which are related to the R_J by

$$R'_{\mathrm{I}} = R_{\mathrm{I}}, \qquad R'_{\mathrm{P}} = R_{\mathrm{P}}, \qquad R'_{\mathrm{T}} = R_{\mathrm{T}} R_{\infty}, \qquad R'_{\mathrm{PT}} = R_{\mathrm{PT}} R_{\infty}. \tag{A.15}$$

Furthermore, we shall need the explicit form of the inner automorphisms

$$\rho_J(R) = R_J R R_J^{-1} \tag{A.16}$$

of $\mathrm{SO}_0(1, 4)$,

$$\rho_{\mathrm{I}}(R) = R = \rho_{\mathrm{T}}(R)$$

$$\rho_{\mathrm{P}}(R) = \begin{pmatrix} R^0{}_0 & -R^0{}_i & R^0{}_5 \\ -R^k{}_0 & R^k{}_i & -R^k{}_5 \\ R^5{}_0 & -R^5{}_i & R^5{}_5 \end{pmatrix}_{k,i=1,2,3} = \rho_{\mathrm{PT}}(R). \tag{A.17}$$

A covering group $\widetilde{\mathrm{SO}}(1, 4)$ of $\mathrm{SO}(1, 4)$ again has four components, the connected component of the identity being equal to G,

$$\widetilde{\mathrm{SO}}(1, 4) = \sum_J G g_J. \tag{A.18}$$

In view of the identity

$$g\, g_J g' g_{J'} = g \tilde{\rho}_J(g') g_J g_{J'} \tag{A.19}$$

and due to $\tilde{\rho}_J(g) = g_J g g_J^{-1} \in G$ the multiplication law of $\widetilde{\mathrm{SO}}(1, 4)$ is completely specified, if the inner automorphisms $\tilde{\rho}_J$ of G and the products $g_J g_{J'}$ are known. In order to determine the $\tilde{\rho}_J$ we note that γ^5, which is not contained in G, maps ξ into

$$\Xi' = (\gamma^5)\Xi(\gamma^5)^{-1}, \qquad \xi' = R'_{\mathrm{PT}}\xi.$$

By defining $\mathscr{E} = \gamma^5 E$, it is easily checked that $\mathscr{E}^{-1} g \mathscr{E} \in G$ and $\tilde{\pi}(\mathscr{E} g \mathscr{E}^{-1}) = \rho_{\mathrm{PT}}(R)$, where $\tilde{\pi}$ denotes the covering homomorphism of $G = \widetilde{\mathrm{SO}}_0(1, 4)$ onto $\mathrm{SO}_0(1, 4)$ (cf (2.10)). Hence we have $\tilde{\rho}_{\mathrm{PT}}(g) = \mathscr{E} g \mathscr{E}^{-1}$ so that the inner automorphisms are obtained to be

$$\tilde{\rho}_{\mathrm{I}}(g) = g = \tilde{\rho}_{\mathrm{T}}(g)$$

$$\tilde{\rho}_{\mathrm{P}}(g) = \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} = \tilde{\rho}_{\mathrm{PT}}(g). \tag{A.20}$$

Furthermore, as in the case of the covering groups of the full Lorentz group (see Grensing 1975a), the multiplication law of the discrete elements reads

$$\begin{array}{c|cccc} & g_{\mathrm{I}} & g_{\mathrm{P}} & g_{\mathrm{T}} & g_{\mathrm{PT}} \\ \hline g_{\mathrm{I}} & g_{\mathrm{I}} & g_{\mathrm{P}} & g_{\mathrm{T}} & g_{\mathrm{PT}} \\ g_{\mathrm{P}} & g_{\mathrm{P}} & \epsilon_{\mathrm{P}} g_{\mathrm{I}} & g_{\mathrm{PT}} & \epsilon_{\mathrm{P}} g_{\mathrm{T}} \\ g_{\mathrm{T}} & g_{\mathrm{T}} & \epsilon_{\mathrm{P}}\epsilon_{\mathrm{T}}\epsilon_{\mathrm{PT}} g_{\mathrm{PT}} & \epsilon_{\mathrm{T}} g_{\mathrm{I}} & \epsilon_{\mathrm{P}}\epsilon_{\mathrm{PT}} g_{\mathrm{P}} \\ g_{\mathrm{PT}} & g_{\mathrm{PT}} & \epsilon_{\mathrm{T}}\epsilon_{\mathrm{PT}} g_{\mathrm{T}} & \epsilon_{\mathrm{T}} g_{\mathrm{P}} & \epsilon_{\mathrm{PT}} g_{\mathrm{I}} \end{array} \qquad \epsilon_{\mathrm{P}}, \epsilon_{\mathrm{T}}, \epsilon_{\mathrm{PT}} = \pm 1. \tag{A.21}$$

On account of

$$g_J g_{J'} = \epsilon g_{J'} g_J, \qquad J \neq J';\, J, J' \neq \mathrm{I} \tag{A.22}$$

with $\epsilon = \epsilon_{\mathrm{P}}\epsilon_{\mathrm{T}}\epsilon_{\mathrm{PT}}$ the discrete elements commute or anti-commute. That is, there exist eight non-isomorphic covering groups of the full de Sitter group.

We apply these results to the field representations (3.1), which we want to extend by the discrete transformations. A simple calculation shows

$$\begin{aligned} \tilde{\rho}_{\mathrm{I}}(g_z) &= g_z = \tilde{\rho}_{\mathrm{T}}(g_z) \\ \tilde{\rho}_{\mathrm{P}}(g_z) &= \mathrm{sgn}(z^0) g_{\Lambda_{\mathrm{P}} z} = \tilde{\rho}_{\mathrm{PT}}(g_z) \end{aligned} \tag{A.23}$$

so that

$$g_z^{-1} g_J g_{\Lambda_J^{-1} z} = \begin{cases} \mathrm{sgn}(z^0)\, g'_{\mathrm{P}} & J = \mathrm{P} \\ g'_{\mathrm{T}} & J = \mathrm{T} \\ \mathrm{sgn}(z^0)\, g'_{\mathrm{PT}} & J = \mathrm{PT} \end{cases} \tag{A.24}$$

where we have defined

$$g'_{\mathrm{I}} = g_{\mathrm{I}}, \qquad g'_{\mathrm{P}} = g_{\mathrm{P}}, \qquad g'_{\mathrm{T}} = g_{\mathrm{T}} g_\infty, \qquad g'_{\mathrm{PT}} = g_{\mathrm{PT}} g_\infty \tag{A.25}$$

in analogy to (A.15). These discrete elements determine the inner automorphisms $\tilde{\rho}'_J(g) = g'_J g g'^{-1}_J$ with

$$\tilde{\rho}'_{\mathrm{P}}(g) = \begin{pmatrix} g_{22} & -g_{21} \\ -g_{12} & g_{11} \end{pmatrix} \qquad \tilde{\rho}'_{\mathrm{T}}(g) = \begin{pmatrix} g_{22} & g_{12} \\ g_{21} & g_{11} \end{pmatrix} \qquad \tilde{\rho}'_{\mathrm{PT}}(g) = \begin{pmatrix} g_{11} & -g_{12} \\ -g_{21} & g_{22} \end{pmatrix}, \tag{A.26}$$

and their multiplication law is specified by

$$\epsilon'_{\mathrm{P}} = +\epsilon_{\mathrm{P}}, \qquad \epsilon'_{\mathrm{T}} = +\epsilon_{\mathrm{T}}, \qquad \epsilon'_{\mathrm{PT}} = -\epsilon_{\mathrm{PT}}. \tag{A.27}$$

In contradistinction to the discrete elements g_J, the g'_J have the properties

$$g'_J \,.\, \mathring{z} = \mathring{z} \tag{A.28}$$

and

$$\tilde{\rho}'(\mathring{g}) = \mathring{g} = \tilde{\rho}'_{\mathrm{PT}}(\mathring{g}), \qquad \tilde{\rho}'_{\mathrm{P}}(\mathring{g}) = \mathring{g}^{\dagger -1} = \tilde{\rho}'_{\mathrm{T}}(\mathring{g}). \tag{A.29}$$

Hence the fixed group of $\mathring{z}$ contains elements $\mathring{g} g'_J$, and the g'_J yield the same inner automorphisms of $\widetilde{\mathrm{SO}}_0(1,3)$ as the corresponding elements of $\widetilde{\mathrm{SO}}(1,3)$. We have now reduced the task to a known problem, because we can make use of the representations of the spin coverings of the full Lorentz group. The representations with dimension less than or equal to four read (Grensing 1975a)

$$s=0: \quad D(\mathring{g})=1; \quad D(g'_{\mathrm{P}})=\hat{\epsilon}_{\mathrm{P}}, \quad D(g'_{\mathrm{T}})=\hat{\epsilon}_{\mathrm{T}} \tag{A.30}$$

$$s=\tfrac{1}{2}: \quad D(\mathring{g})=\mathring{g}; \quad D(g'_{\mathrm{P}})=\gamma^0, \quad D(g'_{\mathrm{T}})=\mathrm{i}\hat{\epsilon}_{\mathrm{T}}\gamma^0\gamma^5 \tag{A.31}$$

$$s=1: \quad D(\mathring{g})=\Lambda; \quad D(g'_{\mathrm{P}})=\epsilon_{\mathrm{P}}\Lambda_{\mathrm{P}}, \quad D(g'_{\mathrm{T}})=\hat{\epsilon}_{\mathrm{T}}\Lambda_{\mathrm{T}} \tag{A.32}$$

with $\hat{\epsilon}_{\mathrm{P}}, \hat{\epsilon}_{\mathrm{T}}=\pm 1$, and where for $s=\frac{1}{2}$ we have restricted ourselves to the case $(\epsilon'_{\mathrm{P}}, \epsilon'_{\mathrm{T}}, \epsilon'_{\mathrm{PT}})=(1,-1,1)$. In addition to (3.1) we thus define

$$T(g'_J)\psi(z)=D(g_z^{-1}g'_J g_{g_J'^{-1}.z})\psi(g_J'^{-1}.z). \tag{A.33}$$

Going over from the g'_J to the g_J, we find

$$T(g_{\mathrm{P}})\psi(z)=D(\mathrm{sgn}(z^0)g'_{\mathrm{P}})\psi(\Lambda_{\mathrm{P}}^{-1}z) \tag{A.34}$$

$$T(g_{\mathrm{T}})\psi(z)=D(g'_{\mathrm{T}})\psi(\Lambda_{\mathrm{T}}^{-1}z)$$

and $T(g_{\mathrm{PT}})=T(g_{\mathrm{P}})T(g_{\mathrm{T}})$ such that for the representations (A.30)–(A.32) the discrete transformations take the form

$$s=0: \quad T(g_{\mathrm{P}})\psi(z)=\hat{\epsilon}_{\mathrm{P}}\psi(\Lambda_{\mathrm{P}}^{-1}z), \quad T(g_{\mathrm{T}})\psi(z)=\hat{\epsilon}_{\mathrm{T}}\psi(\Lambda_{\mathrm{T}}^{-1}z) \tag{A.35}$$

$$s=\tfrac{1}{2}: \quad T(g_{\mathrm{P}})\psi(z)=\mathrm{sgn}(z^0)\gamma^0\psi(\Lambda_{\mathrm{P}}^{-1}z), \quad T(g_{\mathrm{T}})\psi(z)=\mathrm{i}\hat{\epsilon}_{\mathrm{T}}\gamma^0\gamma^5\psi(\Lambda_{\mathrm{T}}^{-1}z) \tag{A.36}$$

$$s=1: \quad T(g_{\mathrm{P}})\psi(z)=\hat{\epsilon}_{\mathrm{P}}\Lambda_{\mathrm{P}}\psi(\Lambda_{\mathrm{P}}^{-1}z), \quad T(g_{\mathrm{T}})\psi(z)=\hat{\epsilon}_{\mathrm{T}}\Lambda_{\mathrm{T}}\psi(\Lambda_{\mathrm{T}}^{-1}z), \tag{A.37}$$

which is the final result. Note that for $s=\frac{1}{2}$ the discrete elements (A.36) commute, though we have induced them with an anti-commuting representation.

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